GEOMETRIC ASPECTS OF MOSER-TRUDINGER INEQUALITIES ON COMPLETE NON-COMPACT RIEMANNIAN MANIFOLDS WITH APPLICATIONS

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Abstract. In this paper we investigate some geometric features of Moser-Trudinger inequalities on complete non-compact Riemannian manifolds. By exploring rearrangement arguments, isoperimetric estimates, and gluing local uniform estimates via Gromov's covering lemma, we characterize the validity of Moser-Trudinger inequalities on complete non-compact $n-$dimensional Riemannian manifolds ($n \geq 2$) with Ricci curvature bounded from below in terms of the volume growth of geodesic balls. These arguments also yield sharp Moser-Trudinger inequalities on Hadamard manifolds which satisfy the Cartan-Hadamard conjecture (e.g., in dimensions 2, 3 and 4). As application, by combining variational arguments, we guarantee the existence of a non-zero isometry-invariant solution for an elliptic problem involving a critical nonlinearity on homogeneous Hadamard manifolds.

1. Introduction and main results

The Moser-Trudinger inequality, as the borderline case of Sobolev inequalities, plays a crucial role in the theory of geometric functional analysis and its applications in the study of quasilinear elliptic problems on the Sobolev space $W^{1,n}$ defined on $n-$dimensional geometric objects, $n \geq 2$.

In the present paper we investigate the influence of geometry of complete non-compact Riemannian manifolds to the validity, sharpness and other aspects of Moser-Trudinger inequalities. Roughly speaking, we shall

- characterize the validity of Moser-Trudinger inequalities on complete non-compact Riemannian manifolds with Ricci curvature bounded from below in terms of the volume growth of geodesic balls (no assumption on the injectivity radius is required);
- provide sharp Moser-Trudinger inequalities on Hadamard manifolds (complete, simply connected Riemannian manifolds with non-positive sectional curvature) whenever the Cartan-Hadamard conjecture holds (e.g., in dimensions 2, 3 and 4);
- guarantee the existence of a non-zero isometry-invariant solution for a quasilinear elliptic problem on $n-$dimensional homogeneous Hadamard manifolds which involves the $n-$Laplace-Beltrami operator and a term with critical growth.

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First of all, in §1.1 we recall some well known features of the Moser-Trudinger inequality which will be used in the sequel. Then, in §1.2 we state and comment our main theoretical results, while in §1.3 we present an application on homogeneous Hadamard manifolds.

1.1. **Facts on Moser-Trudinger inequalities.** Let Ω be an open subset of the Euclidean space $\mathbb{R}^n$ ($n \geq 2$) with finite Lebesgue measure. It is well known that the borderline case of the Sobolev embeddings $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $1 \leq q \leq \frac{np}{n-p}$ and $1 < p < n$, has a pathological behavior; indeed, when $n = p$, the Sobolev space $W_0^{1,n}(\Omega)$ cannot be continuously embedded into $L^\infty(\Omega)$, although formally this should be the case. Motivated by this phenomenon, Trudinger [44] proved that $W_0^{1,n}(\Omega) \hookrightarrow L_{\psi_n}(\Omega)$, where $L_{\psi_n}(\Omega)$ is the Orlicz space associated with the Young function $\psi_n(s) = e^{\alpha|s|^{\frac{n}{n-1}}} - 1$ for $\alpha > 0$ sufficiently small. A few years later, Moser [39] stated the sharp version of this embedding, by proving that there exists $M_0 = M_0(n) > 0$ depending only on n such that

$$\sup_{u\in\mathcal{H}} \int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}} \mathrm{d}x = \begin{cases} M_0 \mathrm{Vol}_e(\Omega) & \text{if} \quad \alpha \in [0, \alpha_n]; \\ +\infty & \text{if} \quad \alpha > \alpha_n; \end{cases} \tag{1.1}$$

here $\mathcal{H} = \left\{u \in W_0^{1,n}(\Omega) : \int_\Omega |\nabla u|^n \mathrm{d}x \leq 1\right\}$, $\mathrm{Vol}_e(\cdot)$ is the Euclidean volume, ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and

$$\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$$

is the *critical exponent*.

The Moser-Trudinger inequality (1.1) became in this way the starting point of further studies in various directions, both in the Euclidean and non-Euclidean settings. In the Euclidean case, milestone results can be found concerning the sharpness and existence of extremal functions for the classical Moser-Trudinger inequality both on bounded and unbounded sets, see e.g. Carleson and Chang [10], Flucher [23], Lin [37], Li and Ruf [36]. In particular, if $n \geq 2$ and

$$\Phi_n(t) = e^t - \sum_{k=0}^{n-2} \frac{t^k}{k!}, \tag{1.2}$$

Li and Ruf [36] proved that

$$S_n^{LR} := \sup_{u\in W^{1,n}(\mathbb{R}^n),\ \|u\|_{0,1}\leq 1} \int_{\mathbb{R}^n} \Phi_n(\alpha_n|u|^{\frac{n}{n-1}})\mathrm{d}x < \infty, \tag{1.3}$$

where $\|u\|_{0,1}^n = \int_{\mathbb{R}^n}(|\nabla u|^n + |u|^n)\mathrm{d}x$. The constant α_n in (1.3) is sharp; although the integral in (1.3) is finite for every $\alpha > 0$ instead of α_n, the supremum is infinite for $\alpha > \alpha_n$. Improvements and higher order extensions of the Moser-Trudinger inequality can be found e.g. in Adams [1], Adimurthi and Druet [2], Cianchi, Lutwak, Yang and Zhang [14], Ibrahim, Masmoudi and Nakanishi [28], Masmoudi and Sani [38], Ruf and Sani [42], and references therein.

Moser-Trudinger inequalities in the non-Euclidean setting captured also special attention. On one hand, sharp Moser-Trudinger inequalities are established in Heisenberg and Carnot groups, see Cohn and Lu [15], Lam and Lu [32], Balogh, Manfredi and Tyson [5], and on CR spheres, see Branson, Fontana and Morpurgo [7]. On the other hand, deep achievements

can be found in the study of Moser-Trudinger inequalities on Riemannian manifolds which are particularly important from the viewpoint of the present paper.

Let $n \geq 2$ and (M, g) be an $n-$dimensional Riemannian manifold endowed with its canonical volume form $\mathrm{d}v_g$. For $\tau > 0$ fixed, on the usual Sobolev space $W^{1,n}(M) = W_0^{1,n}(M)$ defined on (M, g), see Hebey [26], we consider the equivalent norms

$$\|u\|_{0,\tau} = \left(\|\nabla_g u\|^n_{L^n(M)} + \tau^n \|u\|^n_{L^n(M)}\right)^{\frac{1}{n}} \text{ and } \|u\|_{1,\tau} = \|\nabla_g u\|_{L^n(M)} + \tau\|u\|_{L^n(M)},$$

where the Lebesgue norms $\|\cdot\|_{L^n(M)}$ are defined by means of the volume form $\mathrm{d}v_g$. According to these norms, for every $\alpha > 0$, $\tau > 0$ and $i \in \{0, 1\}$, we introduce the quantities

$$S^i_{\alpha,\tau}(M, g) := \sup_{u \in W^{1,n}(M),\ \|u\|_{i,\tau} \leq 1} \int_M \Phi_n(\alpha|u|^{\frac{n}{n-1}})\mathrm{d}v_g,$$

where the function Φ_n comes from (1.2). Since $\|u\|_{0,\tau} \leq \|u\|_{1,\tau}$, then $S^1_{\alpha,\tau}(M, g) \leq S^0_{\alpha,\tau}(M, g)$ for every $\alpha > 0$ and $\tau > 0$.

Let $i \in \{0, 1\}$. If there exists $\alpha > 0$ and $\tau > 0$ such that $S^i_{\alpha,\tau}(M, g) < +\infty$, we say that the *Moser-Trudinger inequality* $(\mathbf{MT})^i_{\alpha,\tau}$ *holds on* (M, g). Contrary, if $S^i_{\alpha,\tau}(M, g) = +\infty$ for some $\alpha > 0$ and $\tau > 0$, we say that the *Moser-Trudinger inequality* $(\mathbf{MT})^i_{\alpha,\tau}$ *fails on* (M, g).

On one hand, when (M, g) is an $n-$dimensional *compact* Riemannian manifold *without boundary*, then for every $\alpha \in [0, \alpha_n]$ and $\tau > 0$, the Moser-Trudinger inequality $(\mathbf{MT})^0_{\alpha,\tau}$ holds on (M, g) and the critical exponent α_n is sharp, see Li [35]; a higher order extension of Li's result can be found in do Ó and Yang [22]. Note that both papers [22] and [35] are extensions of Fontana [24] replacing the constraints $\int_M u\mathrm{d}v_g = 0$ and $\|\nabla_g u\|_{L^n(M)} \leq 1$ from [24] by $\|u\|_{0,\tau} \leq 1$ for every $\tau > 0$. On the other hand, when (M, g) is an $n-$dimensional *compact* Riemannian manifold *with smooth boundary* ∂M, then Cherrier [13] proved that for every $0 \leq \alpha < 2^{\frac{1}{1-n}}\alpha_n$,

$$\sup_{\int_M u\mathrm{d}v_g = 0,\ \|\nabla_g u\|_{L^n(M)} \leq 1} \int_M e^{\alpha|u|^{\frac{n}{n-1}}}\mathrm{d}v_g < \infty, \tag{1.4}$$

and the above constant is sharp, i.e., if $\alpha > 2^{\frac{1}{1-n}}\alpha_n$, then the supremum in (1.4) is infinite.

The study of Moser-Trudinger inequalities on *non-compact* Riemannian manifolds is more delicate, the *curvature* playing a crucial role. On one hand, Yang [47, Theorem 2.3] proved that if (M, g) is an $n-$dimensional complete non-compact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius, then for every $\alpha \in [0, \alpha_n)$, there exists $\tau > 0$ such that $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M, g), while for every $\alpha > \alpha_n$ and $\tau > 0$, $(\mathbf{MT})^1_{\alpha,\tau}$ fails on (M, g). We emphasize that Yang's result deeply exploits the existence of lower bounds on the harmonic radius in terms of bounds on the Ricci curvature and the injectivity radius, see Hebey [26, Theorems 1.2 & 1.3]. On the other hand, by using the arguments from Lam and Lu [32] and fine estimates on the density function of the volume form, Yang, Su and Kong [46] proved that $(\mathbf{MT})^0_{\alpha,\tau}$ holds on every Hadamard manifold (M, g) for every $\alpha \in [0, \alpha_n]$ and $\tau > 0$, and α_n is again sharp; furthermore, as a consequence of Yang [47, Proposition 2.1], the embedding $W^{1,n}(M) \hookrightarrow L^p(M)$ is continuous for every $p \in [n, \infty)$.

1.2. **Main results.** A first observation concerns the failure of Moser-Trudinger inequalities in two different settings without any curvature restriction.

Proposition 1.1. *Let (M,g) be an $n-$dimensional complete Riemannian manifold, $n \geq 2$. The following statements hold*:

(i) *If (M,g) is non-compact with $\mathrm{Vol}_g(M) < \infty$ then for any $\alpha > 0$ and $\tau > 0$, the Moser-Trudinger inequalities $(\mathbf{MT})^i_{\alpha,\tau}$ fail on (M,g), $i \in \{0,1\}$;*

(ii) *For any $\alpha > \alpha_n$ and $\tau > 0$, the Moser-Trudinger inequalities $(\mathbf{MT})^i_{\alpha,\tau}$ fail on (M,g), $i \in \{0,1\}$.*

According to Proposition 1.1, Moser-Trudinger inequalities $(\mathbf{MT})^i_{\alpha,\tau}$ on any $n-$dimensional non-compact complete Riemannian manifold (M,g) are relevant whenever $\mathrm{Vol}_g(M) = \infty$ and the parameter α belongs to the subcritical interval $[0, \alpha_n]$.

Let (M,g) be an $n-$dimensional complete Riemannian manifold and Ω be a smooth open subset in M, $n \geq 2$. We define the *$n-$isoperimetric constant of* Ω as

$$\mathcal{I}_n(\Omega, g) := \inf_A \frac{\mathrm{Area}_g(\partial A)}{\mathrm{Vol}_g(A)^{1-\frac{1}{n}}},$$

where A varies over open sets of Ω having compact closure and smooth boundary. Hereafter, $\mathrm{Area}_g(\partial A)$ stands for the area of ∂A with respect to the metric induced on ∂A by g, and $\mathrm{Vol}_g(A)$ is the volume of A with respect to g. By considering geodesic balls $A := B_x(r)$ in $\Omega \subset M$ with $r \to 0^+$, one clearly has

$$\mathcal{I}_n(\Omega, g) \leq n\omega_n^{\frac{1}{n}}, \tag{1.5}$$

the number $n\omega_n^{\frac{1}{n}}$ being the $n-$dimensional Euclidean isoperimetric constant. For later use, let

$$\mathrm{Isop}(\Omega, g) := \frac{\mathcal{I}_n(\Omega, g)}{n\omega_n^{\frac{1}{n}}} \in [0,1] \tag{1.6}$$

be the *normalized $n-$isoperimetric constant of* Ω.

By using rearrangement arguments on Riemannian manifolds in the spirit of Aubin-Hebey, see [4, 26], we prove the following quantitative result which states a connection between the isoperimetric data of an open set $\Omega \subset M$ and Moser-Trudinger inequalities on (Ω, g):

Theorem 1.1. *Let (M,g) be an $n-$dimensional complete Riemannian manifold, $n \geq 2$, and Ω be a smooth open subset in M such that $\mathrm{Isop}(\Omega, g) > 0$. The following statements hold:*

(i) *If $\mathrm{Vol}_g(\Omega) < \infty$, for any $\alpha \in \left[0, \mathrm{Isop}(\Omega, g)^{\frac{n}{n-1}}\alpha_n\right]$ and $u \in C_0^\infty(\Omega)$ with $\|\nabla_g u\|_{L^n(\Omega)} \leq 1$, one has*
$$\int_\Omega \Phi_n(\alpha|u|^{\frac{n}{n-1}}) \mathrm{d}v_g \leq M_0 \|\nabla_g u\|^n_{L^n(\Omega)} \mathrm{Vol}_g(\Omega),$$
where $M_0 > 0$ is from (1.1).

(ii) *For any $\tau > 0$ and $\alpha \in \left[0, \min\left\{\tau^{\frac{n}{n-1}}, \mathrm{Isop}(\Omega, g)^{\frac{n}{n-1}}\right\}\alpha_n\right]$, the Moser-Trudinger inequalities $(\mathbf{MT})^i_{\alpha,\tau}$ hold on (Ω, g), $i \in \{0,1\}$.*

By exploring Theorem 1.1 (i) and Gromov's covering lemma, we may characterize the validity of Moser-Trudinger inequalities on Riemannian manifolds with Ricci curvature bounded from below:

Theorem 1.2. *Let (M,g) be an $n-$dimensional complete non-compact Riemannian manifold $(n\geq 2)$ with Ricci curvature bounded from below, i.e.,* $\mathrm{Rc}_{(M,g)}\geq kg$ *for some $k\in\mathbb{R}$. Then the following statements are equivalent:*

(i) *There exists $\alpha\in(0,\alpha_n]$ and $\tau>0$ such that $(\mathbf{MT})^0_{\alpha,\tau}$ holds on (M,g);*
(ii) *There exists $\alpha\in(0,\alpha_n]$ and $\tau>0$ such that $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M,g);*
(iii) $\inf_{x\in M}\mathrm{Vol}_g(B_x(1))>0$.

Moreover, any of the above statements imply that the embedding $W^{1,n}(M)\hookrightarrow L^p(M)$ is continuous for every $p\in[n,\infty)$.

Remark 1.1. (a) If (M,g) is an $n-$dimensional complete non-compact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius, it follows by Croke [16] that $\inf_{x\in M}\mathrm{Vol}_g(B_x(1))>0$. Therefore, we may apply Theorem 1.2 in order to prove the validity of $(\mathbf{MT})^1_{\alpha,\tau}$ on (M,g) for *some* $\alpha\in(0,\alpha_n]$ and $\tau>0$, recovering partially the result of Yang [47, Theorem 2.3]. Note that in Yang's result the *positivity of the injectivity radius* is indispensable. Furthermore, our argument shows that once the normalized $n-$isoperimetric constant $\mathrm{Isop}(M,g)$ is close to 1, the value α for which $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M,g) approaches the critical exponent α_n, see Remark 3.1.

(b) Following the approach from Carron [11] and Hebey [26, Lemma 2.2], we stress that the implication (ii)⇒(iii) is valid on *generic* Riemannian manifolds, see Yang [47, Proposition 2.1]. A similar argument also works for (i)⇒(iii).

A remarkable consequence of Theorem 1.2 is as follows:

Corollary 1.1. *Let (M,g) be a two-dimensional complete non-compact Riemannian manifold with non-negative sectional curvature. Then there exists $\alpha\in(0,4\pi]$ and $\tau>0$ such that $(\mathbf{MT})^i_{\alpha,\tau}$ holds on (M,g), $i\in\{0,1\}$.*

Remark 1.2. (a) On one hand, Corollary 1.1 *cannot* be deduced from Yang [47, Theorem 2.3] since no lower bound for the injectivity radius can be guaranteed. Indeed, Croke and Karcher [18] modified the paraboloid of revolution by gluing to it a sequence of disjoint tangential cones in order to obtain a hypersurface with positive sectional curvature and *zero injectivity radius.* On the other hand, under the assumptions of Corollary 1.1 it follows by [18, Theorem A] that

$$\mathrm{Vol}_g(B_x(r))\geq C_M r^2 \ \text{ for every } 0\leq r\leq 1,$$

the constant $C_M\in(0,\pi]$ depending only on (M,g). Thus, it remains to apply Theorem 1.2 to conclude the proof of Corollary 1.1.

(b) We emphasize that Corollary 1.1 is *sharp* with respect to the *dimension.* Indeed, one can construct convex hypersurfaces H in $\mathbb{R}^{n+1}$ with $n\geq 3$, having positive sectional curvature and $\inf_{x\in H}\mathrm{Vol}_g(B_x(1))=0$, see Croke and Karcher [18, p. 755]. Consequently, by Theorem 1.2 and Proposition 1.1 (ii), the Moser-Trudinger inequalities $(\mathbf{MT})^i_{\alpha,\tau}$ fail on H for every $\alpha>0$, $\tau>0$, and $i\in\{0,1\}$.

A hard nut to crack seems to be the following $n-$dimensional volume growth of geodesic balls on Riemannian manifolds with non-negative Ricci curvature:

Question. *Let (M,g) be an $n-$dimensional complete non-compact Riemannian manifold ($n \geq 2$) with non-negative Ricci curvature and assume the Moser-Trudinger inequality $(\mathbf{MT})^0_{\alpha,1}$ holds on (M,g) for some $\alpha \in (0, \alpha_n]$. Is there any $\gamma > 0$ such that*

$$\mathrm{Vol}_g(B_x(r)) \geq \left(\frac{\alpha}{\alpha_n}\right)^\gamma \omega_n r^n \quad \text{for every } x \in M \text{ and } r > 0?$$

Remark 1.3. If the answer is affirmative, we could state that the sharp Moser-Trudinger inequality $(\mathbf{MT})^0_{\alpha_n,1}$ holds on an $n-$dimensional complete non-compact Riemannian manifold (M,g) with non-negative Ricci curvature if and only if (M,g) is isometric to the Euclidean space $\mathbb{R}^n$. Similar results can be found e.g. in do Carmo and Xia [19], Kristály and Ohta [30], Ledoux [34] and references therein for various Sobolev-type inequalities; the arguments in these papers are based on the precise shape of extremal functions for the studied Sobolev-type inequalities in the Euclidean setting. Although Li and Ruf [36] proved that the supremum S_n^{LR} in (1.3) is achieved, no explicit extremal function is known.

As a byproduct of our arguments, we can state sharp Moser-Trudinger inequalities on Hadamard manifolds (complete, simply connected Riemannian manifolds with non-positive sectional curvature) under the validity of the

Cartan-Hadamard conjecture (see Aubin [4]). *Let (M,g) be an $n-$dimensional Hadamard manifold, $n \geq 2$. Then $\mathrm{Isop}(M,g) = 1$, i.e., cf. (1.5), for every bounded open set $A \subset M$ with smooth boundary, one has*

$$\mathrm{Area}_g(\partial A) \geq n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g(A)^{\frac{n-1}{n}}. \tag{1.7}$$

Remark 1.4. Cartan-Hadamard conjecture holds on any Hadamard manifold of dimension 2, cf. Beckenbach and Radó [6] and Weil [45], of dimension 3, cf. Kleiner [29], and of dimension 4, cf. Croke [17], while it is open for dimensions higher than 5.

As a simple consequence of Theorem 1.1 (ii) we can state:

Theorem 1.3. *Let (M,g) be an $n-$dimensional Hadamard manifold ($n \geq 2$) which satisfies the Cartan-Hadamard conjecture. Then for every $\alpha \in [0, \alpha_n]$ and $\tau \geq 1$, the Moser-Trudinger inequalities $(\mathbf{MT})^i_{\alpha,\tau}$ hold on (M,g), $i \in \{0,1\}$. Moreover, the embedding $W^{1,n}(M) \hookrightarrow L^p(M)$ is continuous for every $p \in [n, \infty)$.*

Remark 1.5. In dimensions 2, 3 and 4, Theorem 1.3 coincides with [46, Theorem 1.2] .

1.3. **Application.** We shall present an application of Theorem 1.3 by considering the model elliptic problem

$$-\Delta_{n,g} u + |u|^{n-2}u = f(u) \text{ in } M, \tag{$\mathcal{P}$}$$

where $n \geq 2$, (M,g) is an $n-$dimensional Hadamard manifold, $\Delta_{n,g}u = \mathrm{div}_g(|\nabla_g u|^{n-2}\nabla_g u)$ is the $n-$Laplace-Beltrami operator on (M,g), and the continuous function $f : [0,\infty) \to \mathbb{R}$ satisfies the following hypotheses:

(f_0) there exists $\gamma > n$ such that $f(s) = \mathcal{O}(s^{\gamma-1})$ as $s \to 0^+$;

(f_1) there exists $\alpha_0 > 0$ such that $f(s) = \mathcal{O}(\Phi_n(\alpha_0 s^{\frac{n}{n-1}}))$ as $s \to \infty$, and

$$\lim_{s\to\infty} sf(s)e^{-\alpha_0 s^{\frac{n}{n-1}}} = \infty;$$

(f_2) there exists $\mu > n$ such that $0 < \mu F(s) \leq sf(s)$ for every $s > 0$, where $F(s) = \int_0^s f(t)\mathrm{d}t$;

(f_3) there exist $R_0 > 0$ and $A_0 > 0$ such that $F(s) \leq A_0 f(s)$ for every $s \geq R_0$.

Let $\mathrm{Isom}_g(M)$ be the group of isometries of (M,g) and G be a subgroup of $\mathrm{Isom}_g(M)$. The *orbit of* $x \in M$ under the action of G is $O_G^x = \{\sigma(x) : \sigma \in G\}$. A function $u : M \to \mathbb{R}$ is $G-$*invariant* if $u(\sigma(x)) = u(x)$ for every $x \in M$ and $\sigma \in G$, i.e., u is constant on the orbit O_G^x. The *fixed point set of* G on M is given by $\mathrm{Fix}_M(G) = \{x \in M : \sigma(x) = x \text{ for all } \sigma \in G\}$.

We shall prove the following result:

Theorem 1.4. *Let (M,g) be an $n-$dimensional homogeneous Hadamard manifold ($n \geq 2$), and let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$ and $\mathrm{Card}(O_G^x) = \infty$ for every $x \in M \setminus \{x_0\}$. If $f : [0,\infty) \to \mathbb{R}$ satisfies hypotheses $(f_0) - (f_3)$, then problem $(\mathcal{P})$ has a non-zero, non-negative, $G-$invariant weak solution.*

Remark 1.6. (i) The novelty of Theorem 1.4 is twofold. First, no restriction is imposed on the boundedness from below of the Ricci curvature on (M,g) as in Yang [47, Theorem 2.7]. Second, Theorem 1.4 seems to be the first existence result on non-compact Riemannian manifolds involving exponential terms, by exploring deep features of the isometric group in order to regain some compactness. In order to recover the non-compactness of the space (even in the Euclidean case), instead of the left-hand side of $(\mathcal{P})$, most of the authors considered operators of the form $u \mapsto -\Delta_{n,g}u + V(x)|u|^{n-2}u$ where V is *coercive,* i.e., $V(x) \to \infty$ as $d_g(x_0,x) \to \infty$ for some $x_0 \in M$ fixed, see e.g. Adimurthi and Yang [3], do Ó [20], do Ó and Yang [22], Lam and Lu [31], Yang [47]. Under this coercivity assumption a Rabinowitz-type argument shows that the weighted Sobolev space $W_V^{1,n}(M) = \{u \in W^{1,n}(M) : \int_M V(x)|u|^n \mathrm{d}v_g < \infty\}$ is compactly embedded into $L^p(M)$, $p \in [n,\infty)$. In our case such approach fails. However, in order to prove Theorem 1.4, we shall combine the principle of symmetric criticality of Palais [40] with a recent characterization of compactness of invariant Sobolev spaces à la Lions under the action of isometries, see Skrzypczak and Tintarev [43]. As far as we know, the only result for $V \equiv 1$ in $\mathbb{R}^n$ has been provided recently by do Ó, de Souza, de Medeiros and Severo [21] via a Lions-type concentration-compactness argument.

(ii) Theorem 1.4 is new even in the Euclidean case where one can choose certain subgroups G of the special orthogonal group in $\mathbb{R}^n$. Further examples will be provided in §4 on the $n-$dimensional hyperbolic space, and on the open convex cone of symmetric positive definite matrices endowed with a trace-type scalar product.

(iii) Let $n = 2$ and $f : [0,\infty) \to \mathbb{R}$ be defined by $f(s) = \min\{1,s\}(e^{s^2} - 1)$. Then f satisfies hypotheses $(f_0) - (f_3)$ with $\gamma = \mu = 3$ and $\alpha_0 = R_0 = A_0 = 1$.

2. Preliminaries

In this section we recall those ingredients from Riemannian geometry which will be used throughout the paper. Let (M,g) be an $n-$dimensional Riemannian manifold, T_xM be the tangent space at $x\in M$, $TM=\cup_{x\in M}T_xM$ be the tangent bundle, and $d_g:M\times M\to[0,\infty)$ be the induced metric function by the Riemannian metric g. As usual, let $B_x(r)=\{y\in M: d_g(x,y)<r\}$ and $\overline{B}_x(r)=\{y\in M: d_g(x,y)\leq r\}$ be the open and closed geodesic balls with center $x\in M$ and radius $r>0$, respectively. If $\mathrm{d}v_g$ is the canonical volume element on (M,g), the volume of an open bounded set $\Omega\subset M$ is $\mathrm{Vol}_g(\Omega)=\int_\Omega \mathrm{d}v_g=\mathcal{H}^n(\Omega)$, where $\mathcal{H}^n(S)$ denotes the $n-$dimensional Hausdorff measure of Ω with respect to the metric d_g. Let $\mathrm{d}\sigma_g$ be the $(n-1)-$dimensional Riemann measure induced on $\partial\Omega$ by g; then $\mathrm{Area}_g(\partial\Omega)=\int_{\partial\Omega}\mathrm{d}\sigma_g=\mathcal{H}^{n-1}(\partial\Omega)$ is the area of $\partial\Omega$ with respect to the metric g. For further use, $B_0(\delta)$, $\mathrm{d}x$, $\mathrm{d}\sigma_e$, $\mathrm{Vol}_e(S)$ and $\mathrm{Area}_e(S)$ denote the Euclidean counterparts of the above notions when $S\subset\mathbb{R}^n$.

The behavior of the volume of small geodesic balls can be expressed as follows; for every $x\in M$ we have

$$\mathrm{Vol}_g(B_x(\rho))=\omega_n\rho^n(1+o(\rho)) \text{ as } \rho\to 0, \tag{2.1}$$

see Gallot, Hulin and Lafontaine [25, Theorem 3.98].

The manifold (M,g) has *Ricci curvature bounded from below* if there exists $k\in\mathbb{R}$ such that $\mathrm{Rc}_{(M,g)}\geq kg$ in the sense of bilinear forms, i.e., $\mathrm{Rc}_{(M,g)}(X,X)\geq k|X|_x^2$ for every $X\in T_xM$ and $x\in M$, where $\mathrm{Rc}_{(M,g)}$ is the Ricci curvature, and $|X|_x$ denotes the norm of X with respect to the metric g at the point x. For simplicity of notation, $\langle\cdot,\cdot\rangle_x$ denotes the scalar product g_x on T_xM induced by the metric g. When no confusion arises, if $X,Y\in T_xM$, we simply write $|X|$ and $\langle X,Y\rangle$ instead of $|X|_x$ and $\langle X,Y\rangle_x$, respectively.

In the sequel, $V_k(\rho)$ shall denote the volume of a ball of radius ρ in the $n-$dimensional simply connected, complete Riemannian manifold of constant sectional curvature $k\in\mathbb{R}$. The behavior of the volume of large geodesic balls is given by Bishop-Gromov and Bishop-Gunther:

Proposition 2.1. [25, Theorem 3.101] *Let (M,g) be an $n-$dimensional complete Riemannian manifold. The following statements hold:*

(i) *If $\mathrm{Rc}_{(M,g)}\geq k(n-1)g$ for some $k\in\mathbb{R}$, then $\rho\mapsto\frac{\mathrm{Vol}_g(B_x(\rho))}{V_k(\rho)}$ is non-increasing for every $x\in M$. In particular, by (2.1), one has $\mathrm{Vol}_g(B_x(\rho))\leq V_k(\rho)$ for every $\rho\geq0$ and $x\in M$.*

(ii) *If the sectional curvature of (M,g) is bounded from above by $k\in\mathbb{R}$, then $\mathrm{Vol}_g(B_x(\rho))\geq V_k(\rho)$ for every $\rho\geq0$ and $x\in M$.*

The following result, which is a *local* isoperimetric inequality on Riemannian manifolds with Ricci curvature bounded from below, plays a crucial role in the proof of Theorem 1.2.

Proposition 2.2. [26, Lemma 3.2] *Let (M,g) be an $n-$dimensional complete Riemannian manifold whose Ricci curvature satisfies $\mathrm{Rc}_{(M,g)}\geq kg$ for some $k\in\mathbb{R}$, and suppose that there exists $v>0$ such that $\mathrm{Vol}_g(B_x(1))\geq v$ for every $x\in M$. Then there exist two positive constants $C_0=C(n,k,v)$ and $\eta_0=\eta(n,k,v)$, depending only on $n,k,$ and v, such that for any open set $\Omega\subset M$ with smooth boundary and compact closure, if $\mathrm{Vol}_g(\Omega)\leq\eta_0$, then*

$$\mathrm{Area}_g(\partial\Omega)\geq C_0\mathrm{Vol}_g(\Omega)^{\frac{n-1}{n}}.$$

Gromov's covering lemma, whose proof is based on Proposition 2.1 (i), reads as follows:

Proposition 2.3. [26, Lemma 1.1] *Let (M,g) be an $n-$dimensional complete Riemannian manifold whose Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and let $\rho > 0$ be fixed. Then there exists a sequence $\{x_j\}_{j\in I} \subset M$ (with I countable) such that for every $r \geq \rho$:*

(i) *the family of sets $\{B_{x_j}(r)\}$ is a uniformly locally finite covering of M and there exists an upper bound N_0 for this covering in terms of n, ρ, r and k;*
(ii) *$B_{x_i}(\frac{\rho}{2}) \cap B_{x_j}(\frac{\rho}{2}) = \emptyset$ for every $i \neq j$.*

Let $p \in [1,\infty)$. The norm of $L^p(M)$ is given by $\|u\|_{L^p(M)} = \left(\int_M |u|^p \mathrm{d}v_g\right)^{\frac{1}{p}}$ while $\|\cdot\|_{L^\infty(M)}$ denotes the usual supremum-norm. Let $u : M \to \mathbb{R}$ be a function of class C^1. If (x^i) denotes the local coordinate system on a coordinate neighborhood of $x \in M$, and the local components of the differential of u are $u_i = \frac{\partial u}{\partial x^i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij}u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has

$$|\nabla_g d_g(x_0,\cdot)| = 1 \text{ a.e. on } M. \tag{2.2}$$

The $L^n(M)$ norm of $\nabla_g u(x) \in T_xM$ is given by $\|\nabla_g u\|_{L^n(M)} = \left(\int_M |\nabla_g u|^n \mathrm{d}v_g\right)^{\frac{1}{n}}$, while the space $W^{1,n}(M)$ is the completion of $C_0^\infty(M)$ with respect to the norm $\|\cdot\|_{0,1}$.

In the sequel we adapt the main results from Skrzypczak and Tintarev [43] to our setting concerning the Sobolev spaces in the presence of group-symmetries; for a similar approach see also Hebey and Vaugon [27]. When (M,g) is a Hadamard manifold, the embedding $W^{1,n}(M) \hookrightarrow L^p(M)$ is continuous for every $p \in [n,\infty)$ (cf. Theorem 1.3), but not compact. By exploiting the fact that the embedding $W^{1,n}(M) \hookrightarrow L^p(M)$ is (weakly) cocompact relative to the isometry group $\mathrm{Isom}_g(M)$ for every $p \in (n,\infty)$, one can state the following result:

Proposition 2.4. [43, Theorem 1.3 & Proposition 3.1] *Let (M,g) be an $n-$dimensional homogeneous Hadamard manifold and G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G)$ is a singleton. Then the subspace of $G-$invariant functions of $W^{1,n}(M)$, i.e., $W^{1,n}_G(M) = \{u \in W^{1,n}(M) : u \circ \sigma = u \text{ for all } \sigma \in G\}$ is compactly embedded into $L^p(M)$ for every $p \in (n,\infty)$.*

We conclude this section with the principle of symmetric criticality of Palais [40]. A group G *acts continuously* on a real Banach space W by an application $[\sigma,u] \mapsto \sigma u$ from $G \times W$ to W if this map itself is continuous on $G \times W$ and

- $\sigma_{\mathrm{id}} u = u$ for every $u \in W$, where $\sigma_{\mathrm{id}} \in G$ is the identity element of G;
- $(\sigma_1\sigma_2)u = \sigma_1(\sigma_2 u)$ for every $\sigma_1, \sigma_2 \in G$ and $u \in W$;
- $u \mapsto \sigma u$ is linear for every $\sigma \in G$.

Proposition 2.5. [40] *Let W be a real Banach space, G be a compact topological group acting continuously on W by a map $[\sigma,u] \mapsto \sigma u$ from $G \times W$ to W, and $h : W \to \mathbb{R}$ be a $G-$invariant C^1-function, i.e., $h(\sigma u) = h(u)$ for every $(\sigma,u) \in G \times W$. If $u_G \in \mathrm{Fix}_W(G) = \{u \in W : \sigma u = u \text{ for all } \sigma \in G\}$ is a critical point of $h_G = h|_{\mathrm{Fix}_W(G)}$, then u_G is also a critical point of h.*

3. Proof of main results

Proof of Proposition 1.1. (i) It is enough to prove the statement for $(\mathbf{MT})^1_{\alpha,\tau}$. By contradiction, let us assume that the Moser-Trudinger inequality $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M,g) for some $\alpha>0$ and $\tau>0$. Due to Yang [47, Proposition 2.1], there exists $v>0$ such that $\mathrm{Vol}_g(B_x(1))\geq v$ for every $x\in M$. A similar argument as in Hebey [26] based on Zorn lemma shows that there exists a sequence $\{x_i\}_{i\in I}\subset M$ such that the balls $B_{x_i}(1)$ and $B_{x_j}(1)$ are disjoint for every $i\neq j$ and $M=\bigcup_{i\in I}B_{x_i}(2)$. Note that

$$+\infty>\mathrm{Vol}_g(M)\geq\sum_{i\in I}\mathrm{Vol}_g(B_{x_i}(1))\geq \mathrm{Card}(I)v.$$

Therefore, I is finite, which implies together with the Hopf-Rinow theorem that M is covered by a finite number of relatively compact sets; thus M is compact, a contradiction.

(ii) Similar statement for $(\mathbf{MT})^1_{\alpha,\tau}$ is presented by Yang [47] on Riemannian manifolds with Ricci curvature bounded from below and positive injectivity radius. In fact, since the Moser-type truncation functions are locally constructed, the proof in [47] works in generic Riemannian manifolds as well; for completeness we provide its proof since some parts will be used later on.

Let $x_0\in M$ be arbitrarily fixed and denote by i_{x_0} the injectivity radius at x_0; clearly, $i_{x_0}>0$. Choose also $\varepsilon_0\in(0,i_{x_0})$ sufficiently small such that it belongs to the range of (2.1). For every $\varepsilon\in(0,\varepsilon_0)$, we introduce the Moser-type truncation function $u_\varepsilon:M\to[0,\infty)$ defined by

$$u_\varepsilon(x)=\min\left\{\left(\frac{\log\frac{\varepsilon_0}{d_g(x_0,x)}}{\log\frac{\varepsilon_0}{\varepsilon}}\right)_+,1\right\}, \tag{3.1}$$

where $r_+=\max\{0,r\}$ for $r\in\mathbb{R}$. The functions u_ε can be approximated by elements from $C_0^\infty(M)$ and we shall see that $u_\varepsilon\in W^{1,n}(M)$.

Indeed, on one hand, due to (2.1), (2.2) and the layer cake representation, one has

$$\begin{aligned}
\|\nabla_g u_\varepsilon\|^n_{L^n(M)} &= \int_M|\nabla_g u_\varepsilon|^n\mathrm{d}v_g\\
&= \left(\log\frac{\varepsilon_0}{\varepsilon}\right)^{-n}\int_{B_{x_0}(\varepsilon_0)\setminus B_{x_0}(\varepsilon)}d_g(x_0,x)^{-n}\mathrm{d}v_g(x)\\
&= \left(\log\frac{\varepsilon_0}{\varepsilon}\right)^{-n}\left(\frac{\mathrm{Vol}_g(B_{x_0}(\varepsilon_0))}{\varepsilon_0^n}-\frac{\mathrm{Vol}_g(B_{x_0}(\varepsilon))}{\varepsilon^n}+n\int_\varepsilon^{\varepsilon_0}\mathrm{Vol}_g(B_{x_0}(\rho))\rho^{-1-n}\mathrm{d}\rho\right)\\
&= \left(\log\frac{\varepsilon_0}{\varepsilon}\right)^{-n}\left(\mathcal{O}(1)+n\omega_n\int_\varepsilon^{\varepsilon_0}\rho^{-1}(1+o(\rho))\mathrm{d}\rho\right)\\
&= n\omega_n\left(\log\frac{\varepsilon_0}{\varepsilon}\right)^{1-n}\left(1+\mathcal{O}\left(\left(\log\frac{\varepsilon_0}{\varepsilon}\right)^{-1}\right)\right),
\end{aligned}$$

as $\varepsilon \to 0$. On the other hand, again by the layer cake representation and (2.1), we obtain that

$$\begin{aligned}
\|u_\varepsilon\|^n_{L^n(M)} &= \int_M u_\varepsilon^n \mathrm{d}v_g \\
&= \mathrm{Vol}_g(B_{x_0}(\varepsilon)) + \Big(\log\frac{\varepsilon_0}{\varepsilon}\Big)^{-n} \int_{B_{x_0}(\varepsilon_0)\setminus B_{x_0}(\varepsilon)} \left(\log \frac{\varepsilon_0}{d_g(x_0,x)}\right)^n \mathrm{d}v_g(x) \\
&= \mathrm{Vol}_g(B_{x_0}(\varepsilon)) + n\Big(\log\frac{\varepsilon_0}{\varepsilon}\Big)^{-n} \int_\varepsilon^{\varepsilon_0} (\mathrm{Vol}_g(B_{x_0}(\rho)) - \mathrm{Vol}_g(B_{x_0}(\varepsilon))) \left(\log\frac{\varepsilon_0}{\rho}\right)^{n-1} \frac{\mathrm{d}\rho}{\rho} \\
&= n\Big(\log\frac{\varepsilon_0}{\varepsilon}\Big)^{-n} \int_\varepsilon^{\varepsilon_0} \mathrm{Vol}_g(B_{x_0}(\rho)) \left(\log\frac{\varepsilon_0}{\rho}\right)^{n-1} \frac{\mathrm{d}\rho}{\rho} \\
&= \mathcal{O}\left(\Big(\log\frac{\varepsilon_0}{\varepsilon}\Big)^{-n}\right),
\end{aligned}$$

as $\varepsilon \to 0$. Consequently, since $n\omega_n = \omega_{n-1}$, if $\tau > 0$ is arbitrarily fixed, one has that

$$\|u_\varepsilon\|_{1,\tau} = \omega_{n-1}^{\frac{1}{n}} \Big(\log\frac{\varepsilon_0}{\varepsilon}\Big)^{\frac{1-n}{n}} \left(1 + \mathcal{O}\left(\Big(\log\frac{\varepsilon_0}{\varepsilon}\Big)^{-\frac{1}{n}}\right)\right). \tag{3.2}$$

Therefore, if $\alpha > \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, by relations (2.1) and (3.2) it follows that

$$\begin{aligned}
S^1_{\alpha,\tau}(M,g) &\geq \lim_{\varepsilon\to 0} \int_M \Phi_n\left(\alpha \frac{u_\varepsilon^{\frac{n}{n-1}}}{\|u_\varepsilon\|_{1,\tau}^{\frac{n}{n-1}}}\right) \mathrm{d}v_g \geq \lim_{\varepsilon\to 0} \int_{B_{x_0}(\varepsilon)} \Phi_n\left(\frac{\alpha}{\|u_\varepsilon\|_{1,\tau}^{\frac{n}{n-1}}}\right) \mathrm{d}v_g \\
&= \lim_{\varepsilon\to 0} \left(\mathrm{Vol}_g(B_{x_0}(\varepsilon)) \Phi_n\left(\alpha \|u_\varepsilon\|_{1,\tau}^{\frac{n}{1-n}}\right)\right) \\
&= \lim_{\varepsilon\to 0} \left(\mathrm{Vol}_g(B_{x_0}(\varepsilon)) \left(e^{\alpha\|u_\varepsilon\|_{1,\tau}^{\frac{n}{1-n}}} - \sum_{k=0}^{n-2} \frac{\alpha^k \|u_\varepsilon\|_{1,\tau}^{\frac{kn}{1-n}}}{k!}\right)\right) \\
&= \varepsilon_0^{\alpha\omega_{n-1}^{\frac{1}{1-n}}} \lim_{\varepsilon\to 0} \varepsilon^{n-\alpha\omega_{n-1}^{\frac{1}{1-n}}} \\
&= +\infty,
\end{aligned}$$

which means that the Moser-Trudinger inequality $(\mathbf{MT})^1_{\alpha,\tau}$ fails on (M,g). Since $S^1_{\alpha,\tau}(M,g) \leq S^0_{\alpha,\tau}(M,g)$, $(\mathbf{MT})^0_{\alpha,\tau}$ also fails on (M,g), which concludes the proof. □

Proof of Theorem 1.1. We divide the proof into four steps.

Step 1: choice of test functions. Since for every $u \in W^{1,n}(M)$ we have $|\nabla_g u| = |\nabla_g |u||$ a.e. on M, classical Morse theory and density argument show that Moser-Trudinger inequalities on (M,g) are sufficient to be considered for continuous test functions $u : M \to [0,\infty)$ having compact support $S \subset M$, where S is enough smooth, u being of class C^∞ in S and having only non-degenerate critical points in S.

Step 2: Pólya-Szegő-type inequality. Let $\Omega \subset M$ be an open set and we consider a non-negative function $u \in C_0^\infty(\Omega)$ with the properties from Step 1. To this function u, we associate

its Euclidean rearrangement function $u^* : \mathbb{R}^n \to [0,\infty)$ which is radially symmetric, non-increasing in $|x|$, and for every $t>0$ is defined by

$$\mathrm{Vol}_e(\{x\in\mathbb{R}^n : u^*(x)>t\}) = \mathrm{Vol}_g(\{x\in\Omega : u(x)>t\}). \tag{3.3}$$

It is clear by (3.3) that

$$\mathrm{Vol}_g(\Omega)\geq \mathrm{Vol}_g(S) = \mathrm{Vol}_g(\mathrm{supp}(u)) = \mathrm{Vol}_e(\mathrm{supp}(u^*)) = \mathrm{Vol}_e(B_0(R_S)) \tag{3.4}$$

for some $R_S>0$. By the layer cake representation, for every $q>0$ one has

$$\begin{aligned}\int_\Omega u^q \mathrm{d}v_g &= \int_0^\infty \mathrm{Vol}_g(\{x\in\Omega : u^q(x)>t\})\mathrm{d}t = \int_0^\infty \mathrm{Vol}_e(\{x\in\mathbb{R}^n : (u^*)^q(x)>t\}) \\ &= \int_{B_0(R_S)} (u^*)^q \mathrm{d}x. \end{aligned}\tag{3.5}$$

For abbreviation, we consider the sets

$$A_t = \{x\in\Omega : u(x)>t\},\quad A_t^* = \{x\in\mathbb{R}^n : u^*(x)>t\}$$

for every $0<t<\|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)}$. The boundaries of A_t and A_t^* are exactly the level sets $\partial A_t = u^{-1}(t)\subset S\subset\Omega$ and $\partial A_t^* = (u^*)^{-1}(t)\subset\mathbb{R}^n$, which are regular. Since u^* is radially symmetric, the set ∂A_t^* is an $(n-1)-$dimensional sphere for every $0<t<\|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)}$. Therefore, by relation (3.3) we have that

$$\begin{aligned}\mathrm{Area}_g(\partial A_t) &\geq \mathcal{I}_n(\Omega,g)\mathrm{Vol}_g(A_t)^{\frac{n-1}{n}} = \mathcal{I}_n(\Omega,g)\mathrm{Vol}_e(A_t^*)^{\frac{n-1}{n}} \\ &= \mathrm{Isop}(\Omega,g)\mathrm{Area}_e(\partial A_t^*).\end{aligned}\tag{3.6}$$

Let

$$V(t) = \mathrm{Vol}_g(A_t) = \mathrm{Vol}_e(A_t^*).$$

The co-area formula (see Chavel [12, pp. 302-303]) gives

$$V'(t) = -\int_{\partial A_t}\frac{1}{|\nabla_g u|}\mathrm{d}\sigma_g = -\int_{\partial A_t^*}\frac{1}{|\nabla u^*|}\mathrm{d}\sigma_e. \tag{3.7}$$

Since $|\nabla u^*|$ is constant on the sphere ∂A_t^*, by (3.7) one has that

$$V'(t) = -\frac{\mathrm{Area}_e(\partial A_t^*)}{|\nabla u^*(x)|},\ x\in\Gamma_t^*. \tag{3.8}$$

By (3.7) again and Hölder's inequality it turns out that

$$\mathrm{Area}_g(\partial A_t) = \int_{\partial A_t}\mathrm{d}\sigma_g \leq (-V'(t))^{\frac{n-1}{n}}\left(\int_{\partial A_t}|\nabla_g u|^{n-1}\mathrm{d}\sigma_g\right)^{\frac{1}{n}}.$$

For every $0 < t < \|u\|_{L^\infty(M)}$, by using (3.6) and (3.8), it follows that

$$\begin{aligned}\int_{\partial A_t} |\nabla_g u|^{n-1} \mathrm{d}\sigma_g &\geq \mathrm{Area}_g(\partial A_t)^n \left(-V'(t)\right)^{1-n} \\ &\geq \mathrm{Isop}(\Omega, g)^n \mathrm{Area}_e(\partial A_t^*)^n \left(\frac{\mathrm{Area}_e(\partial A_t^*)}{|\nabla u^*(x)|}\right)^{1-n} \qquad (x \in \Gamma_t^*) \\ &= \mathrm{Isop}(\Omega, g)^n \int_{\partial A_t^*} |\nabla u^*|^{n-1} \mathrm{d}\sigma_e.\end{aligned}$$

The co-area formula and the above estimate give a Pólya-Szegő-type inequality

$$\begin{aligned}\int_\Omega |\nabla_g u|^n \mathrm{d}v_g &= \int_0^\infty \int_{\partial A_t} |\nabla_g u|^{n-1} \mathrm{d}\sigma_g \mathrm{d}t \\ &\geq \mathrm{Isop}(\Omega, g)^n \int_0^\infty \int_{\partial A_t^*} |\nabla u^*|^{n-1} \mathrm{d}\sigma_e \\ &= \mathrm{Isop}(\Omega, g)^n \int_{\mathbb{R}^n} |\nabla u^*|^n \mathrm{d}x \\ &= \mathrm{Isop}(\Omega, g)^n \int_{B_0(R_S)} |\nabla u^*|^n \mathrm{d}x. \qquad (3.9)\end{aligned}$$

Now, we are ready to prove the claims (i) and (ii).

Step 3: proof of (i). Let Ω be an open subset of M such that $\mathrm{Isop}(\Omega, g) > 0$, $\mathrm{Vol}_g(\Omega) < \infty$ and let $u \in C_0^\infty(\Omega)$ be a non-negative and non-zero function with the properties from Step 1 and

$$\|\nabla_g u\|_{L^n(\Omega)} \leq 1. \qquad (3.10)$$

Let $\tilde{u} = \frac{u}{\|\nabla_g u\|_{L^n(\Omega)}}$. Applying the arguments from Step 2 for the function $\tilde{u}$, by (3.9) it follows that

$$1 = \|\nabla_g \tilde{u}\|_{L^n(\Omega)} \geq \mathrm{Isop}(\Omega, g) \|\nabla \tilde{u}^*\|_{L^n(B_0(R_S))}, \qquad (3.11)$$

where $\tilde{u}^*$ is the Euclidean rearrangement function of $\tilde{u}$. Thus, for every $\alpha \in \left[0, \mathrm{Isop}(\Omega, g)^{\frac{n}{n-1}} \alpha_n\right]$, we have

$$\begin{aligned}\int_\Omega \Phi_n\left(\alpha |u|^{\frac{n}{n-1}}\right) \mathrm{d}v_g &= \sum_{j=n-1}^\infty \frac{\alpha^j}{j!} \int_\Omega \tilde{u}^{\frac{nj}{n-1}} \|\nabla_g u\|_{L^n(\Omega)}^{\frac{nj}{n-1}} \mathrm{d}v_g \\ &\leq \|\nabla_g u\|_{L^n(\Omega)}^n \sum_{j=n-1}^\infty \frac{\alpha^j}{j!} \int_\Omega \tilde{u}^{\frac{nj}{n-1}} \mathrm{d}v_g \qquad \text{[see (3.10)]} \\ &= \|\nabla_g u\|_{L^n(\Omega)}^n \sum_{j=n-1}^\infty \frac{\alpha^j}{j!} \int_{B_0(R_S)} (\tilde{u}^*)^{\frac{nj}{n-1}} \mathrm{d}x \qquad \text{[see (3.5)]}\end{aligned}$$

$$
\begin{aligned}
&= \|\nabla_g u\|^n_{L^n(\Omega)} \int_{B_0(R_S)} \Phi_n\left(\alpha(\tilde u^*)^{\frac{n}{n-1}}\right)\mathrm{d}x \\
&\leq \|\nabla_g u\|^n_{L^n(\Omega)} \int_{B_0(R_S)} \Phi_n\left(\alpha_n(\mathrm{Isop}(\Omega,g)\tilde u^*)^{\frac{n}{n-1}}\right)\mathrm{d}x \\
&\leq M_0\|\nabla_g u\|^n_{L^n(\Omega)}\mathrm{Vol}_e(B_0(R_S)) \qquad \text{[see (3.11) and (1.1)]} \\
&\leq M_0\|\nabla_g u\|^n_{L^n(\Omega)}\mathrm{Vol}_g(\Omega). \qquad \text{[see (3.4)]}
\end{aligned}
$$

<u>*Step 4*</u>: *proof of* (ii). Let us fix $\tau > 0$ and $\alpha \in \left[0, \min\{\tau^{\frac{n}{n-1}}, \mathrm{Isop}(\Omega,g)^{\frac{n}{n-1}}\}\alpha_n\right]$, and let $u \in C_0^\infty(\Omega)$ be a non-negative and non-zero function with the properties from Step 1 and $\|u\|_{0,\tau} \leq 1$. Then, by (3.9) we have

$$
\begin{aligned}
1 &\geq \|u\|_{0,\tau} = \left(\int_\Omega (|\nabla_g u|^n + \tau^n u^n)\mathrm{d}v_g\right)^{1/n} \\
&\geq \left(\int_{\mathbb{R}^n} (\mathrm{Isop}(\Omega,g))^n|\nabla u^*|^n + \tau^n(u^*)^n)\mathrm{d}x\right)^{1/n} \\
&\geq \min\{\mathrm{Isop}(\Omega,g),\tau\}\|u^*\|_{0,1}.
\end{aligned}
$$

By this estimate, (3.5) and (1.3) one has

$$
\begin{aligned}
\int_\Omega \Phi_n\left(\alpha|u|^{\frac{n}{n-1}}\right)\mathrm{d}v_g &= \int_{\mathbb{R}^n} \Phi_n\left(\alpha(u^*)^{\frac{n}{n-1}}\right)\mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \Phi_n\left(\alpha_n(\min\{\mathrm{Isop}(\Omega,g),\tau\}u^*)^{\frac{n}{n-1}}\right)\mathrm{d}x \\
&\leq S_n^{LR},
\end{aligned}
$$

i.e., $(\mathbf{MT})^0_{\alpha,\tau}$ holds on (Ω,g). This fact also implies that $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (Ω,g). □

Proof of Theorem 1.2. (i)⇔(ii) is trivial since the two norms $\|\cdot\|_{0,\tau}$ and $\|\cdot\|_{1,\tau}$ are equivalent. More precisely, if $(\mathbf{MT})^0_{\alpha,\tau}$ holds on (M,g) for some $\alpha > 0$ and $\tau > 0$, then $(\mathbf{MT})^1_{\alpha,\tau}$ also holds on (M,g); conversely, if $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M,g) then $(\mathbf{MT})^0_{\tilde\alpha,\tau}$ holds on (M,g) where $\tilde\alpha = 2^{\frac{n}{1-n}}\alpha$.

(ii)⇒(iii) is given in Yang [47, Proposition 2.1] for generic Riemannian manifolds; namely, if $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M,g) for some $\alpha > 0$ and $\tau > 0$ then for every $q > n$, $x \in M$ and $r > 0$, one has

$$
\mathrm{Vol}_g(B_x(r)) \geq \min\left\{\frac{1}{2\tau Q}, \frac{r}{2^{\frac{2q-n}{q-n}}Q}\right\}^{\frac{nq}{q-n}},
$$

where Q depends on n, q, α and $S^1_{\alpha,\tau}(M,g) < \infty$.

(iii)⇒(ii) Let $v > 0$ be such that $\mathrm{Vol}_g(B_x(1)) \geq v$ for every $x \in M$. According to Proposition 2.2, there exist two positive constants $C_0 = C(n,k,v)$ and $\eta_0 = \eta(n,k,v)$, depending only on

n, k and v, such that for any open set $\Omega \subset M$ with smooth boundary and compact closure, if $\mathrm{Vol}_g(\Omega) \leq \eta_0$, then $\mathrm{Area}_g(\partial\Omega) \geq C_0 \mathrm{Vol}_g(\Omega)^{\frac{n-1}{n}}$. Consequently, one has

$$\mathrm{Isop}(\Omega, g) \geq \frac{C_0}{n\omega_n^{\frac{1}{n}}} \quad \text{for all smooth open set } \Omega \subset M \text{ and } \mathrm{Vol}_g(\Omega) \leq \eta_0. \tag{3.12}$$

Let $\rho_0 > 0$ be small enough such that $\omega_n \rho_0^n e^{\sqrt{(n-1)|k|}\rho_0} \leq \eta_0$. According to Proposition 2.1 (i), it turns out that

$$\mathrm{Vol}_g(B_x(\rho_0)) \leq \eta_0 \text{ for all } x \in M. \tag{3.13}$$

By Proposition 2.3, there exists a sequence $\{x_j\}_{j\in\mathbb{N}} \subset M$ such that $B_{x_i}(\frac{\rho_0}{4}) \cap B_{x_j}(\frac{\rho_0}{4}) = \emptyset$ for every $i \neq j$ and the family of geodesic balls $B_{x_j}(\frac{\rho_0}{2})$ is a uniformly locally finite covering of M, the number $N_0 \in \mathbb{N}$ being the uniform upper bound for this covering (which depends only on ρ_0, n and k). Let us fix $x_0 \in M$ arbitrarily. For every $j \in \mathbb{N}$, let

$$\psi_j(x) = \min\left\{\left(2 - \frac{2}{\rho_0} d_g(x_j, x)\right)_+, 1\right\},\ x \in M.$$

We have that $\psi_j \in W^{1,n}(M)$, $0 \leq \psi_j \leq 1$, $\psi_j(x) = 1$ for every $x \in B_{x_j}(\frac{\rho_0}{2})$, $\psi_j(x) = 0$ for every $x \in M \setminus B_{x_j}(\rho_0)$, while $|\nabla_g \psi_j(x)| = \frac{2}{\rho_0}$ for a.e. $x \in B_{x_j}(\rho_0) \setminus B_{x_j}(\frac{\rho_0}{2})$ (cf. (2.2)) and $|\nabla_g \psi_j(x)| = 0$ otherwise. The uniform upper bound for the above covering yields that

$$1 \leq \sum_{j\in\mathbb{N}} \psi_j(x) \leq N_0 \ \text{ for all } x \in M. \tag{3.14}$$

Let $\tau = \frac{4}{\rho_0}$ and fix $u \in C_0^\infty(M)$ arbitrarily such that $\|u\|_{1,\tau} \leq 1$. By the latter relation and the properties of ψ_j we have for every $j \in \mathbb{N}$ that

$$\begin{aligned}
\|\nabla_g(\psi_j^2 u)\|_{L^n(M)} &= \|\psi_j^2 \nabla_g u + u \nabla_g \psi_j^2\|_{L^n(M)} \leq \|\psi_j^2 \nabla_g u\|_{L^n(M)} + 2\|u\psi_j \nabla_g \psi_j\|_{L^n(M)} \\
&\leq \|\nabla_g u\|_{L^n(M)} + \tau\|u\|_{L^n(M)} = \|u\|_{1,\tau} \\
&\leq 1.
\end{aligned}$$

This estimate and relations (3.13) and (3.12) show that for every $j \in \mathbb{N}$ we can apply Theorem 1.1(i) to the geodesic ball $B_{x_j}(\rho_0)$ and function $\psi_j^2 u$ (standard density arguments allow to consider that $\psi_j^2 u$ is smooth), obtaining for every $\alpha \in \left[0, (C_0^n n^{-n} \omega_n^{-1})^{\frac{1}{n-1}} \alpha_n\right]$ that

$$\int_{B_{x_j}(\rho_0)} \Phi_n(\alpha|\psi_j^2 u|^{\frac{n}{n-1}}) \mathrm{d}v_g \leq M_0 \eta_0 \|\nabla_g(\psi_j^2 u)\|^n_{L^n(B_{x_j}(\rho_0))}. \tag{3.15}$$

By the properties of the function ψ_j and the covering of M, it follows that

$$\begin{aligned}
\int_M \Phi_n(\alpha|u|^{\frac{n}{n-1}})\mathrm{d}v_g &\leq \sum_{j\in\mathbb{N}}\int_{B_{x_j}(\frac{\rho_0}{2})}\Phi_n(\alpha|u|^{\frac{n}{n-1}})\mathrm{d}v_g \leq \sum_{j\in\mathbb{N}}\int_{B_{x_j}(\rho_0)}\Phi_n(\alpha|\psi_j^2 u|^{\frac{n}{n-1}})\mathrm{d}v_g \\
&\leq M_0\eta_0\sum_{j\in\mathbb{N}}\|\nabla_g(\psi_j^2 u)\|^n_{L^n(B_{x_j}(\rho_0))} \qquad \text{[see (3.15)]} \\
&= M_0\eta_0\sum_{j\in\mathbb{N}}\|\psi_j^2\nabla_g u + u\nabla_g\psi_j^2\|^n_{L^n(B_{x_j}(\rho_0))} \\
&\leq M_0\eta_0 2^n\left(\sum_{j\in\mathbb{N}}\int_{B_{x_j}(\rho_0)}\psi_j|\nabla_g u|^n\mathrm{d}v_g + \frac{4^n}{\rho_0^n}\sum_{j\in\mathbb{N}}\int_{B_{x_j}(\rho_0)}\psi_j|u|^n\mathrm{d}v_g\right) \\
&\leq M_0\eta_0 2^n N_0\left(\int_M|\nabla_g u|^n\mathrm{d}v_g + \frac{4^n}{\rho_0^n}\int_M|u|^n\mathrm{d}v_g\right) \qquad \text{[see (3.14)]} \\
&= M_0\eta_0 2^n N_0\|u\|^n_{0,\tau} \\
&\leq M_0\eta_0 2^n N_0. \qquad \text{[since } \|u\|_{0,\tau}\leq\|u\|_{1,\tau}\leq 1]
\end{aligned}$$

Consequently, $S^1_{\alpha,\tau}(M,g)\leq M_0\eta_0 2^n N_0$ for $\tau=\frac{4}{\rho_0}$ and every $\alpha\in\left[0,(C_0^n n^{-n}\omega_n^{-1})^{\frac{1}{n-1}}\alpha_n\right]$, where the constants M_0, C_0, η_0, N_0 and ρ_0 depend only on n, k and v.

The continuity of the embedding $W^{1,n}(M)\hookrightarrow L^p(M)$ for every $p\in[n,\infty)$ follows as in Yang [47, Proposition 2.1] whenever any of the assumptions (i), (ii) or (iii) holds. $\square$

Remark 3.1. If $\mathrm{Isop}(M,g)$ is close to 1, the constant C_0 in (3.12) can be fixed close to $n\omega_n^{\frac{1}{n}}$. In such a case, the latter proof shows that those numbers $\alpha\geq 0$ for which the Moser-Trudinger inequality $(\mathbf{MT})^1_{\alpha,\tau}$ holds on (M,g) approaches the critical exponent α_n.

Proof of Theorem 1.3. If the Cartan-Hadamard conjecture holds on (M,g), then $\mathrm{Isop}(M,g)=1$. Now, if $\alpha\in[0,\alpha_n]$ and $\tau\geq 1$, it remains to apply Theorem 1.1 (ii). The continuity of the embedding $W^{1,n}(M)\hookrightarrow L^p(M)$ for every $p\in[n,\infty)$ follows again by Yang [47, Proposition 2.1]. $\square$

Remark 3.2. Let (M,g) be an $n-$dimensional Hadamard manifold, $n\geq 2$. Precisely as in the Euclidean case, one can prove:

$$\Phi_n(\alpha|u|^{\frac{n}{n-1}})\in L^1(M) \ \text{ for all } \alpha>0,\ u\in W^{1,n}(M). \tag{3.16}$$

The proof of (3.16) is based on the validity of the Moser-Trudinger inequality $(\mathbf{MT})^0_{\alpha,\tau}$ on (M,g) for some $\alpha>0$ and $\tau\geq 1$ (cf. Theorem 1.3), the density of $C_0^\infty(M)$ in $W^{1,n}(M)$ endowed with the norm $\|\cdot\|_{0,\tau}$, and basic properties of the function Φ_n; a similar argument on Riemannian manifolds with Ricci curvature bounded from below is presented in Yang [47, p. 1911].

4. Application: Proof of Theorem 1.4

Without mentioning explicitly, we assume throughout this section that all assumptions of Theorem 1.4 are satisfied. By (f_0), one has that $f(0) = 0$; therefore, we extend continuously the function $f : [0, \infty) \to \mathbb{R}$ to the whole $\mathbb{R}$ by $f(s) = 0$ for $s \leq 0$; thus, $F(s) = 0$ for $s \leq 0$ as well. The function $u \in W^{1,n}(M)$ is a *weak solution* of problem $(\mathcal{P})$ if

$$\int_M (|\nabla_g u|^{n-2}\langle \nabla_g u, \nabla_g w\rangle + |u|^{n-2}uw)\mathrm{d}v_g = \int_M f(u)w\mathrm{d}v_g \ \text{ for all } w \in W^{1,n}(M). \tag{4.1}$$

By the above extension it turns out that every weak solution of problem $(\mathcal{P})$ is non-negative.

Let $\mathcal{E} : W^{1,n}(M) \to \mathbb{R}$ be the energy functional associated with problem $(\mathcal{P})$, given by

$$\mathcal{E}(u) = \frac{\|u\|_{0,1}^n}{n} - \mathcal{F}(u),$$

where

$$\mathcal{F}(u) = \int_M F(u)\mathrm{d}v_g.$$

Due to (f_0), (f_1), there exists $c_0 > 0$ such that

$$|f(s)| \leq c_0 \left(|s|^{\gamma-1} + \Phi_n(\alpha_0|s|^{\frac{n}{n-1}})\right) \ \text{ for all } s \in \mathbb{R}. \tag{4.2}$$

Therefore, by hypothesis (f_2), Hölder's inequality and the inequality

$$\Phi_n(s)^q \leq \Phi_n(qs) \text{ for every } q \geq 1, \tag{4.3}$$

it follows for every $u \in W^{1,n}(M)$ that

$$\begin{aligned} 0 \leq \mathcal{F}(u) &\leq c_0 \int_M |u|^\gamma \mathrm{d}v_g + c_0 \int_M |u|\Phi_n(\alpha_0|u|^{\frac{n}{n-1}})\mathrm{d}v_g \\ &\leq c_0\|u\|_{L^\gamma(M)}^\gamma + c_0\|u\|_{L^n(M)} \left(\int_M \Phi_n\left(\frac{\alpha_0 n}{n-1}|u|^{\frac{n}{n-1}}\right)\mathrm{d}v_g\right)^{\frac{n-1}{n}}. \end{aligned}$$

The continuous embedding $W^{1,n}(M) \hookrightarrow L^p(M)$ for every $p \in [n, \infty)$ and relation (3.16) imply that the latter term in the above estimate is finite, i.e., the energy functional $\mathcal{E}$ is well-defined on $W^{1,n}(M)$; furthermore, $\mathcal{E}$ is of class C^1 on $W^{1,n}(M)$ and standard arguments yield that the critical points of $\mathcal{E}$ are precisely the weak solutions of problem $(\mathcal{P})$.

Let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ with the required properties, i.e., $\mathrm{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$ and $\mathrm{Card}(O_G^x) = \infty$ for every $x \in M \setminus \{x_0\}$. The action of G on $W^{1,n}(M)$ is defined by

$$(\sigma u)(x) = u(\sigma^{-1}(x)) \ \text{ for all } \sigma \in G, \ u \in W^{1,n}(M), \ x \in M, \tag{4.4}$$

where $\sigma^{-1} : M \to M$ is the inverse of the isometry σ. Let

$$W_G^{1,n}(M) = \{u \in W^{1,n}(M) : \sigma u = u \text{ for all } \sigma \in G\}$$

be the subspace of $G-$invariant functions of $W^{1,n}(M)$ and let $\mathcal{E}_G : W_G^{1,n}(M) \to \mathbb{R}$ be the restriction of the energy functional $\mathcal{E}$ to $W_G^{1,n}(M)$.

Several lemmas are needed in order to complete the proof of Theorem 1.4.

Lemma 4.1. *Every critical point of $\mathcal{E}_G$ is a non-negative $G-$invariant weak solution of $(\mathcal{P})$.*

Proof. We first notice that G acts continuously on $W^{1,n}(M)$ by relation (4.4); for instance, for every $\sigma_1, \sigma_2 \in G$, $u \in W^{1,n}(M)$ and $x \in M$ one has

$$((\sigma_1\sigma_2)u)(x) = u((\sigma_1\sigma_2)^{-1}(x)) = u(\sigma_2^{-1}(\sigma_1^{-1}(x))) = (\sigma_2 u)(\sigma_1^{-1}(x)) = (\sigma_1(\sigma_2 u))(x),$$

while the other properties trivially hold.

We claim that $\mathcal{E}$ is $G-$invariant. To see this, let $u \in W^{1,n}(M)$ and $\sigma \in G$ be arbitrarily fixed. Since $\sigma : M \to M$ is an isometry on M, by (4.4), for every $x \in M$ we have

$$\nabla_g(\sigma u)(x) = D\sigma_{\sigma^{-1}(x)}\nabla_g u(\sigma^{-1}(x)),$$

where $D\sigma_{\sigma^{-1}(x)} : T_{\sigma^{-1}(x)}M \to T_x M$ denotes the differential of σ at the point $\sigma^{-1}(x)$. Note that the (signed) Jacobian determinant of σ is 1 and $D\sigma_{\sigma^{-1}(x)}$ preserves inner products. Therefore, by using the latter facts, relation (4.4) and a change of variables $y = \sigma^{-1}(x)$, it turns out that

$$\begin{aligned}
\|\sigma u\|_{0,1}^n &= \int_M (|\nabla_g(\sigma u)(x)|_x^n + |(\sigma u)(x)|^n)\,\mathrm{d}v_g(x) \\
&= \int_M \left(|\nabla_g u(\sigma^{-1}(x))|_{\sigma^{-1}(x)}^n + |u(\sigma^{-1}(x))|^n\right)\mathrm{d}v_g(x) = \int_M \left(|\nabla_g u(y)|_y^n + |u(y)|^n\right)\mathrm{d}v_g(y) \\
&= \|u\|_{0,1}^n,
\end{aligned}$$

and

$$\mathcal{F}(\sigma u) = \int_M F((\sigma u)(x))\mathrm{d}v_g(x) = \int_M F(u(\sigma^{-1}(x)))\mathrm{d}v_g(x) = \int_M F(u(y))\mathrm{d}v_g(y) = \mathcal{F}(u),$$

which ends the proof of the claim.

Note that $\mathrm{Fix}_{W^{1,n}(M)}(G)$ is nothing but $W_G^{1,n}(M)$; therefore, if $u_G \in W_G^{1,n}(M)$ is a critical point of $\mathcal{E}_G$, then due to Proposition 2.5, u_G is also a critical point of $\mathcal{E}$ and as such, u_G turns out to be a $G-$invariant non-negative weak solution of $(\mathcal{P})$, as we pointed out before. □

Lemma 4.2. *The functional $\mathcal{E}_G$ has the mountain pass geometry, i.e.,*

(i) *for every non-negative, compactly supported $\tilde{u} \in W_G^{1,n}(M) \setminus \{0\}$ we have $\mathcal{E}_G(s\tilde{u}) \to -\infty$ as $s \to \infty$;*

(ii) *there exist $\tilde{r} > 0$ and $\tilde{\delta} > 0$ such that $\mathcal{E}_G(u) \geq \tilde{\delta}$ for every $u \in W_G^{1,n}(M)$ with $\|u\|_{0,1} = \tilde{r}$.*

Proof. (i) Let $\tilde{u} \in W_G^{1,n}(M) \setminus \{0\}$ be a non-negative function with compact support contained in the geodesic ball $\overline{B}_{x_0}(r)$ for some $r > 0$. By (f_2), it follows that there exist $c_1, c_2 > 0$ such that $F(t) \geq c_1 t^\mu - c_2$ for every $t \in [0, \infty)$. Therefore,

$$\mathcal{E}_G(s\tilde{u}) = s^n \frac{\|\tilde{u}\|_{0,1}^n}{n} - \mathcal{F}(s\tilde{u}) \leq s^n \frac{\|\tilde{u}\|_{0,1}^n}{n} - c_1 s^\mu \int_{\overline{B}_{x_0}(r)} \tilde{u}^\mu \mathrm{d}v_g + c_2 \mathrm{Vol}_g(\overline{B}_{x_0}(r)).$$

Since $\tilde{u} \neq 0$ and $\mu > n$, one has that $\mathcal{E}_G(s\tilde{u}) \to -\infty$ as $s \to \infty$.

(ii) By (f_0) and (f_1), there exists $c_3 > 0$ such that

$$|f(s)| \leq c_3 |s|^{\gamma - 1}\left(1 + \Phi_n(\alpha_0 |s|^{\frac{n}{n-1}})\right) \text{ for all } s \in \mathbb{R}. \tag{4.5}$$

By Hölder's inequality and (4.3), for every $u \in W_G^{1,n}(M)$ one has

$$\begin{aligned}\mathcal{F}(u) &\leq c_3\|u\|_{L^\gamma(M)}^\gamma + c_3\int_M |u|^\gamma \Phi_n(\alpha_0|u|^{\frac{n}{n-1}})\mathrm{d}v_g \\ &\leq c_3\|u\|_{L^\gamma(M)}^\gamma + c_3\|u\|_{L^{2\gamma}(M)}^\gamma \left(\int_M \Phi_n(2\alpha_0|u|^{\frac{n}{n-1}})\mathrm{d}v_g\right)^{\frac{1}{2}}. \end{aligned} \tag{4.6}$$

Due to [46, Theorem 1.2] (or Theorem 1.3 in dimensions 2, 3 and 4), the Moser-Trudinger inequality $(\mathbf{MT})_{\alpha_n,1}^0$ is valid on (M,g), i.e., $S_{\alpha_n,1}^0(M,g) < \infty$. Let $\mathfrak{s}_p > 0$ be the best embedding constant in $W^{1,n}(M) \hookrightarrow L^p(M)$, $p \in [n,\infty)$, and let us choose $\tilde r > 0$ such that

$$2\alpha_0\tilde r^{\frac{n}{n-1}} \leq \alpha_n \text{ and } c_3 n\left(\mathfrak{s}_\gamma^\gamma + \mathfrak{s}_{2\gamma}^\gamma (S_{\alpha_n,1}^0(M,g))^{\frac{1}{2}}\right)\tilde r^{\gamma-n} < 1. \tag{4.7}$$

Thus, for every $u \in W_G^{1,n}(M)$ with $\|u\|_{0,1} = \tilde r$, by relations (4.6) and (4.7) it follows that

$$\mathcal{E}_G(u) \geq \frac{\tilde r^n}{n} - c_3\left(\mathfrak{s}_\gamma^\gamma + \mathfrak{s}_{2\gamma}^\gamma (S_{\alpha_n,1}^0(M,g))^{\frac{1}{2}}\right)\tilde r^\gamma := \tilde\delta > 0,$$

which concludes the proof. $\square$

The next lemma gives information on the behavior of Palais-Smale sequences of the functional $\mathcal{E}_G$; let $W_G^{1,n}(M)^*$ be the dual of $W_G^{1,n}(M)$, and $\langle\cdot,\cdot\rangle_*$ be the duality pairing between $W_G^{1,n}(M)^*$ and $W_G^{1,n}(M)$.

Lemma 4.3. *If $\{u_j\}_{j\in\mathbb{N}} \subset W_G^{1,n}(M)$ is a Palais-Smale sequence of $\mathcal{E}_G$, i.e., $\mathcal{E}_G(u_j) \to c \in \mathbb{R}$ and $\mathcal{E}_G'(u_j) \to 0$ in $W_G^{1,n}(M)^*$, then there exist a subsequence of $\{u_j\}$ (still denoted by $\{u_j\}$) and $u_G \in W_G^{1,n}(M)$ such that*

(i) $\lim_{j\to\infty}\mathcal{F}(u_j) = \mathcal{F}(u_G)$;
(ii) $u_j \to u_G$ strongly in $L^p(M)$ for every $p \in (n,\infty)$;
(iii) $\mathcal{E}_G'(u_G) = 0$, *i.e., u_G is a critical point of $\mathcal{E}_G$.*

Proof. (i)&(ii) Let $\{u_j\}_{j\in\mathbb{N}} \subset W_G^{1,n}(M)$ be a Palais-Smale sequence of $\mathcal{E}_G$ at level $c \in \mathbb{R}$, i.e., $\mathcal{E}_G(u_j) \to c$ and $|\langle \mathcal{E}_G'(u_j), w\rangle_*| \leq \varepsilon_j\|w\|_{0,1}$ for every $w \in W_G^{1,n}(M)$, where $\lim_{j\to\infty}\varepsilon_j = 0$; explicitly, one has

$$\frac{\|u_j\|_{0,1}^n}{n} - \mathcal{F}(u_j) \to c; \tag{4.8}$$

$$\left|\int_M (|\nabla_g u_j|^{n-2}\langle\nabla_g u_j, \nabla_g w\rangle + |u_j|^{n-2}u_j w)\mathrm{d}v_g - \int_M f(u_j)w\mathrm{d}v_g\right| \leq \varepsilon_j\|w\|_{0,1}, \forall w \in W_G^{1,n}(M). \tag{4.9}$$

By construction, $f(s) = F(s) = 0$ for $s \leq 0$; thus, multiplying relation (4.8) by μ, letting $w = u_j$ in (4.9), and adding these relations, it follows by hypothesis (f_2) that

$$\left(\frac{\mu}{n} - 1\right)\|u_j\|_{0,1}^n \leq \int_M (\mu F(u_j) - f(u_j)u_j)\mathrm{d}v_g + \mu|c| + \varepsilon_j\|u_j\|_{0,1} \leq \mu|c| + \varepsilon_j\|u_j\|_{0,1}.$$

Since $\mu > n$, the sequence $\{u_j\}$ is bounded in $W_G^{1,n}(M)$; in particular, by relation (4.8) and the latter estimate one can guarantee the existence of $c_4 > 0$ (depending only on n, μ and c) such that for every $j \in \mathbb{N}$,

$$\mathcal{F}(u_j) = \int_M F(u_j)\mathrm{d}v_g \leq c_4 \quad \text{and} \quad \int_M f(u_j)u_j\mathrm{d}v_g \leq c_4. \tag{4.10}$$

By the boundedness of $\{u_j\}$ in $W_G^{1,n}(M)$ together with the hypothesis $\mathrm{Fix}_M(G) = \{x_0\}$ and Proposition 2.4, there exists $u_G \in W_G^{1,n}(M)$ such that, up to a subsequence, we have

$$u_j \rightharpoonup u_G \text{ weakly in } W_G^{1,n}(M); \tag{4.11}$$

$$u_j \to u_G \text{ strongly in } L^p(M) \text{ for every } p \in (n,\infty); \tag{4.12}$$

$$u_j \to u_G \text{ a.e. in } M. \tag{4.13}$$

Let $\varepsilon > 0$ be fixed arbitrarily, and let

$$K > \max\left\{R_0, \frac{A_0}{\varepsilon}c_4, \frac{A_0}{\varepsilon}\int_M f(u_G)u_G\mathrm{d}v_g\right\}, \tag{4.14}$$

where $R_0 > 0$ and $A_0 > 0$ are from (f_3). Since $F(s) = 0$ for every $s \in (-\infty, 0]$ and $f(s)s \geq 0$ for every $s \in [0,\infty)$ (cf. (f_2)), by hypothesis (f_3) and relations (4.14) and (4.10), one has for every $j \in \mathbb{N}$ that

$$\begin{aligned}
\int_{\{|u_j|>K\}} F(u_j)\mathrm{d}v_g &= \int_{\{u_j>K\}} F(u_j)\mathrm{d}v_g \leq A_0 \int_{\{u_j>K\}} f(u_j)\mathrm{d}v_g \\
&\leq \frac{A_0}{K}\int_{\{u_j>K\}} f(u_j)u_j\mathrm{d}v_g \leq \frac{A_0}{K}c_4 \\
&< \varepsilon.
\end{aligned} \tag{4.15}$$

In a similar way, we have

$$\int_{\{|u_G|>K\}} F(u_G)\mathrm{d}v_g \leq A_0\int_{\{u_G>K\}} f(u_G)\mathrm{d}v_g \leq \frac{A_0}{K}\int_{\{u_G>K\}} f(u_G)u_G\mathrm{d}v_g < \varepsilon. \tag{4.16}$$

By relation (4.5), it follows that $f(s) \leq c_3 s^{\gamma-1}\left(1+\Phi_n(\alpha_0 K^{\frac{n}{n-1}})\right)$ for all $s \in [0,K]$. Therefore,

$$F(s) \leq c_5 s^\gamma \text{ for all } s \in [0,K],$$

where $c_5 = c_3\left(1+\Phi_n(\alpha_0 K^{\frac{n}{n-1}})\right)$. Consequently, for every $j \in \mathbb{N}$ we have

$$\chi_{\{|u_j|\leq K\}}F(u_j) \leq c_5|u_j|^\gamma, \tag{4.17}$$

where χ_A denotes the characteristic function of the set $A \subset M$. We recall the inequality

$$||s|^p - |t|^p| \leq p|s-t|(|s|^{p-1} + |t|^{p-1}) \text{ for all } p > 1 \text{ and } t, s \in \mathbb{R}. \tag{4.18}$$

By (4.18) and Hölder's inequality, one has

$$\begin{aligned}
\int_M ||u_j|^\gamma - |u_G|^\gamma|\,\mathrm{d}v_g &\leq \gamma\int_M |u_j - u_G|(|u_j|^{\gamma-1} + |u_G|^{\gamma-1})\mathrm{d}v_g \\
&\leq \gamma\|u_j - u_G\|_{L^\gamma(M)}(\|u_j\|_{L^\gamma(M)}^{\gamma-1} + \|u_G\|_{L^\gamma(M)}^{\gamma-1}).
\end{aligned}$$

Since $\gamma > n$, due to (4.12) the latter term tends to zero, thus $|u_j|^\gamma$ converges to $|u_G|^\gamma$ in $L^1(M)$ as $j \to \infty$. By (4.13), (4.17) and the generalized Lebesgue dominated convergence theorem we have

$$\lim_{j\to\infty} \int_M \chi_{\{|u_j|\leq K\}} F(u_j) \mathrm{d}v_g = \int_M \chi_{\{|u_G|\leq K\}} F(u_G) \mathrm{d}v_g.$$

The latter relation together with (4.15) and (4.16) implies that

$$\lim_{j\to\infty} \int_M F(u_j) \mathrm{d}v_g = \int_M F(u_G) \mathrm{d}v_g,$$

which proves (i). Note that (4.12) is precisely the property (ii).

(iii) The proof is divided into several steps.

Step 1:

$$\lim_{j\to\infty} \int_M f(u_j) w \mathrm{d}v_g = \int_M f(u_G) w \mathrm{d}v_g \text{ for all } w \in C_0^\infty(M). \tag{4.19}$$

This step is similar to (i); let $\varepsilon > 0$ and $w \in C_0^\infty(M) \setminus \{0\}$ be arbitrarily fixed, and let

$$K > \frac{\|w\|_{L^\infty(M)}}{\varepsilon} \max\left\{c_4, \int_M f(u_G) u_G \mathrm{d}v_g\right\}.$$

Relation (4.10), the choice of $K > 0$ and the fact that $|f(s)s| = f(s)s$ for every $s \in \mathbb{R}$ show that

$$\int_{\{|u_j|>K\}} |f(u_j)w| \, \mathrm{d}v_g < \varepsilon \quad \text{and} \quad \int_{\{|u_G|>K\}} |f(u_G)w| \, \mathrm{d}v_g < \varepsilon. \tag{4.20}$$

As above, by (4.5), one has $f(s) \leq c_3 s^{\gamma-1}\left(1 + \Phi_n(\alpha_0 K^{\frac{n}{n-1}})\right)$ for all $s \in [0, K]$. Therefore,

$$\chi_{\{|u_j|\leq K\}} |f(u_j)w| \leq c_6 |u_j|^{\gamma-1} |w|, \tag{4.21}$$

where $c_6 = c_3\left(1 + \Phi_n(\alpha_0 K^{\frac{n}{n-1}})\right)$, which is formally the same as c_5 but perhaps K differs. Note that $|u_j|^{\gamma-1}|w|$ converges to $|u_G|^{\gamma-1}|w|$ in $L^1(M)$; indeed, since $\gamma > n \geq 2$, by (4.18) and Hölder's inequality we have

$$\begin{aligned}\int_M \left||u_j|^{\gamma-1} - |u_G|^{\gamma-1}\right| |w| \mathrm{d}v_g &\leq (\gamma-1) \int_M |u_j - u_G| (|u_j|^{\gamma-2} + |u_G|^{\gamma-2}) |w| \mathrm{d}v_g \\ &\leq (\gamma-1) \|u_j - u_G\|_{L^\gamma(M)} (\|u_j\|^{\gamma-2}_{L^\gamma(M)} + \|u_G\|^{\gamma-2}_{L^\gamma(M)}) \|w\|_{L^\gamma(M)},\end{aligned}$$

and according to (4.12), the above integral tends to zero as $j \to \infty$. The generalized Lebesgue dominated convergence theorem together with (4.13) and (4.21) provide

$$\lim_{j\to\infty} \int_M \chi_{\{|u_j|\leq K\}} f(u_j) w \mathrm{d}v_g = \int_M \chi_{\{|u_G|\leq K\}} f(u_G) w \mathrm{d}v_g.$$

Combining the latter relation with (4.20), the claim (4.19) follows.

Step 2: *for every compact set $S \subset M \setminus \{x_0\}$, one has*

$$\lim_{j\to\infty} \int_S |f(u_j)(u_j - u_G)| \mathrm{d}v_g = 0. \tag{4.22}$$

In order to prove this claim, let $\delta_0 > 0$ be fixed such that

$$\alpha_0 \frac{\gamma}{\gamma - 1} 2^{\frac{n+1}{n-1}} \delta_0^{\frac{1}{n-1}} < \alpha_n, \tag{4.23}$$

where $\gamma > n$ and $\alpha_0 > 0$ are from hypotheses (f_0) and (f_1), respectively. We are going to prove first an energy-concentration property; namely, we claim that for every $x \in M \setminus \{x_0\}$ there exists $0 < r_x < d_g(x_0, x)$ such that

$$\lim_{j\to\infty} \int_{B_x(r_x)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g < \delta_0. \tag{4.24}$$

By contradiction, we assume that there exists $\tilde{x} \in M \setminus \{x_0\}$ such that

$$\lim_{r\to 0} \lim_{j\to\infty} \int_{B_{\tilde{x}}(r)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g \geq \delta_0.$$

By assumption, we have $\mathrm{Card}(O_G^{\tilde{x}}) = \infty$; thus, we may fix the distinct points $\tilde{x}_1, ..., \tilde{x}_N \in O_G^{\tilde{x}}$ with

$$N > \frac{n(|c| + c_4)}{\delta_0},$$

where $c \in \mathbb{R}$ and $c_4 > 0$ are from (4.8) and (4.10), respectively. Note that there exists $\sigma_l \in G$ such that $\tilde{x}_l = \sigma_l(\tilde{x})$ for every $l \in \{1, ..., N\}$. Furthermore, $B_{\tilde{x}_l}(r) = \sigma_l B_{\tilde{x}}(r)$ for every $l \in \{1, ..., N\}$. By using these facts, since u_j are $G-$invariant functions and $\sigma_l \in G$ are isometries on M, a similar argument as in the proof of Lemma 4.1 shows that for every $l \in \{1, ..., N\}$,

$$\int_{B_{\tilde{x}_l}(r)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g = \int_{\sigma_l B_{\tilde{x}}(r)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g = \int_{B_{\tilde{x}}(r)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g.$$

By relations (4.8), (4.10) and the above assumption, it follows that

$$\begin{aligned} n(|c| + c_4) &\geq \lim_{j\to\infty} \|u_j\|_{0,1}^n = \lim_{j\to\infty} \int_M (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g \\ &\geq \sum_{l=1}^N \lim_{r\to 0} \lim_{j\to\infty} \int_{B_{\tilde{x}_l}(r)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g = N \lim_{r\to 0} \lim_{j\to\infty} \int_{B_{\tilde{x}}(r)} (|\nabla_g u_j|^n + |u_j|^n) \mathrm{d}v_g \\ &\geq N\delta_0, \end{aligned}$$

which contradicts the choice of N. Therefore, relation (4.24) holds.

Let $x \in M \setminus \{x_0\}$ be arbitrarily fixed, $r := r_x > 0$ from (4.24) and $a_j = \frac{1}{\mathrm{Vol}_g(B_x(r))} \int_{B_x(r)} u_j \mathrm{d}v_g$. By Hölder's inequality and (4.24), for enough large $j \in \mathbb{N}$ we have

$$|a_j| \leq \frac{1}{\mathrm{Vol}_g(B_x(r))} \int_{B_x(r)} |u_j| \mathrm{d}v_g \leq \mathrm{Vol}_g(B_x(r))^{-\frac{1}{n}} \left(\int_{B_x(r)} |u_j|^n \mathrm{d}v_g \right)^{\frac{1}{n}} \leq \left(\frac{\delta_0}{\mathrm{Vol}_g(B_x(r))} \right)^{\frac{1}{n}}.$$

Let $\tilde{u}_j = u_j - a_j$ for every $j \in \mathbb{N}$. Then for enough large $j \in \mathbb{N}$, one has

$$\int_{B_x(r)} \tilde{u}_j \mathrm{d}v_g = 0 \quad \text{and} \quad \int_{B_x(r)} |\nabla_g \tilde{u}_j|^n \mathrm{d}v_g < \delta_0.$$

Therefore, by relation (4.23) and Cherrier's result (cf. (1.4)) applied on $\overline{B}_x(r)$ for the functions $\frac{\tilde{u}_j}{\|\nabla_g \tilde{u}_j\|_{L^n(B_x(r))}}$, $j \in \mathbb{N}$ large enough, it follows that

$$\begin{aligned}
\int_{B_x(r)} e^{\alpha_0 \frac{\gamma}{\gamma-1}|u_j|^{\frac{n}{n-1}}} \mathrm{d}v_g &= \int_{B_x(r)} e^{\alpha_0 \frac{\gamma}{\gamma-1}|\tilde{u}_j + a_j|^{\frac{n}{n-1}}} \mathrm{d}v_g \\
&\leq e^{\alpha_0 \frac{\gamma}{\gamma-1} 2^{\frac{n}{n-1}} |a_j|^{\frac{n}{n-1}}} \int_{B_x(r)} e^{\alpha_0 \frac{\gamma}{\gamma-1} 2^{\frac{n}{n-1}} |\tilde{u}_j|^{\frac{n}{n-1}}} \mathrm{d}v_g \\
&\leq c_7,
\end{aligned}$$

where the constant $c_7 > 0$ depends on α_0, n, r, x, γ and δ_0, but not on $j \in \mathbb{N}$.

Since $\{u_j\}$ is bounded in $L^\gamma(M)$, the latter estimate together with Hölder's inequality and relations (4.2) and (4.3) yield

$$\begin{aligned}
I_j &:= \int_{B_x(r)} |f(u_j)(u_j - u_G)| \mathrm{d}v_g \\
&\leq \left(\int_{B_x(r)} |f(u_j)|^{\frac{\gamma}{\gamma-1}} \mathrm{d}v_g \right)^{1-\frac{1}{\gamma}} \left(\int_{B_x(r)} |u_j - u_G|^\gamma \mathrm{d}v_g \right)^{\frac{1}{\gamma}} \\
&\leq 2c_0 \left(\int_{B_x(r)} |u_j|^\gamma \mathrm{d}v_g + \int_{B_x(r)} \Phi_n \left(\alpha_0 \frac{\gamma}{\gamma - 1} |u_j|^{\frac{n}{n-1}} \right) \mathrm{d}v_g \right)^{1-\frac{1}{\gamma}} \|u_j - u_G\|_{L^\gamma(B_x(r))} \\
&\leq c_8 \|u_j - u_G\|_{L^\gamma(M)},
\end{aligned}$$

where $c_8 > 0$ does not depend on $j \in \mathbb{N}$. Consequently, due to (4.12), we have

$$\lim_{j\to\infty} I_j = 0.$$

Now, the compact set $S \subset M \setminus \{0\}$ can be covered by a finite number of geodesic balls with the above properties, which completes the proof of (4.22) throughout the latter limit.

Step 3: *for every compact set $S \subset M \setminus \{x_0\}$, one has*

$$\lim_{j\to\infty} \int_S \left(|\nabla_g u_j - \nabla_g u_G|^n + |u_j - u_G|^n \right) \mathrm{d}v_g = 0. \tag{4.25}$$

Let $x \in M \setminus \{x_0\}$ be arbitrarily fixed and $r := r_x < d_g(x_0, x)$ from (4.24). For every $0 < \rho \leq r$, let $A_{x_0}(\rho) = B_{x_0}(d_g(x_0, x) + \rho) \setminus \overline{B}_{x_0}(d_g(x_0, x) - \rho)$ be the open geodesic annulus with center $x_0 \in M$ and radii $d_g(x_0, x) \pm \rho$, respectively.

We consider a $d_g(x_0, \cdot)-$radially symmetric function $\varphi \in C_0^\infty(A_{x_0}(r))$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $A_{x_0}(\frac{r}{2})$. Hereafter, a function $\varphi : M \to \mathbb{R}$ is called $d_g(x_0, \cdot)-$radially symmetric, if there exists a function $h_\varphi : [0, \infty) \to \mathbb{R}$ such that $\varphi(x) = h_\varphi(d_g(x_0, x))$ for every $x \in M$. For simplicity, we extend φ by zero to the whole M outside of the geodesic annulus $A_{x_0}(r)$.

Note that φ is $G-$invariant. Indeed, since $\mathrm{Fix}_M(G) = \{x_0\}$, for every $x \in M$ and isometry $\sigma \in G$ we have

$$\varphi(\sigma(x)) = h_\varphi(d_g(x_0, \sigma(x))) = h_\varphi(d_g(\sigma(x_0), \sigma(x))) = h_\varphi(d_g(x_0, x)) = \varphi(x).$$

In particular, $\varphi(u_j-u_G)\in W_G^{1,n}(M)$ for every $j\in\mathbb{N}$; inserting this test-function into (4.9), we obtain

$$\int_M |\nabla_g u_j|^{n-2}\langle \nabla_g u_j,(u_j-u_G)\nabla_g\varphi+\varphi(\nabla_g u_j-\nabla_g u_G)\rangle {\rm d}v_g+\int_M \varphi|u_j|^{n-2}u_j(u_j-u_G){\rm d}v_g$$
$$-\int_M \varphi f(u_j)(u_j-u_G){\rm d}v_g\leq \varepsilon_j\|\varphi(u_j-u_G)\|_{0,1}.$$

Reorganizing this inequality, it yields that

$$\begin{aligned}
J_j \ &:= \ \int_{A_{x_0}(r)} \varphi\langle |\nabla_g u_j|^{n-2}\nabla_g u_j-|\nabla_g u_G|^{n-2}\nabla_g u_G,\nabla_g u_j-\nabla_g u_G\rangle {\rm d}v_g\\
&\quad +\int_{A_{x_0}(r)}\varphi\left(|u_j|^{n-2}u_j-|u_G|^{n-2}u_G\right)(u_j-u_G){\rm d}v_g\\
&\leq \ \int_{A_{x_0}(r)}(u_G-u_j)|\nabla_g u_j|^{n-2}\langle \nabla_g u_j,\nabla_g\varphi\rangle {\rm d}v_g+\int_{A_{x_0}(r)}\varphi|\nabla_g u_G|^{n-2}\langle \nabla_g u_G,\nabla_g u_G-\nabla_g u_j\rangle {\rm d}v_g\\
&\quad +\int_{A_{x_0}(r)}\varphi|u_G|^{n-2}u_G(u_G-u_j){\rm d}v_g+\int_{A_{x_0}(r)}\varphi f(u_j)(u_j-u_G){\rm d}v_g+\varepsilon_j\|\varphi(u_j-u_G)\|_{0,1}.
\end{aligned}$$

We shall check that every term on the right hand side of the above inequality tend to 0 as $j\to\infty$. First, by Hölder's inequality, we have

$$\left|\int_{A_{x_0}(r)}(u_j-u_G)|\nabla_g u_j|^{n-2}\langle \nabla_g u_j,\nabla_g\varphi\rangle {\rm d}v_g\right|\leq$$
$$\leq \|u_j-u_G\|_{L^\gamma(M)}{\rm Vol_g}(A_{x_0}(r))^{\frac{\gamma-n}{\gamma n}}\|\nabla_g u_j\|^{n-1}_{L^n(M)}\|\nabla_g\varphi\|_{L^\infty(M)}.$$

Since $\{u_j\}$ is bounded in $W_G^{1,n}(M)$ and $\gamma>n$, due to (4.12), the latter expression tends to 0 as $j\to\infty$. Second, due to (4.11), one has in particular that $\nabla_g u_j \rightharpoonup \nabla_g u_G$ weakly in $L^n(A_{x_0}(r),TM)$. Therefore,

$$\lim_{j\to\infty}\int_{A_{x_0}(r)}\langle\varphi|\nabla_g u_G|^{n-2}\nabla_g u_G,\nabla_g u_j-\nabla_g u_G\rangle {\rm d}v_g=0.$$

The third term trivially converges to 0. Due to (4.22), the fourth term tends to 0 as well. Since $\{\varphi(u_j-u_G)\}$ is bounded in $W_G^{1,n}(M)$ and $\lim_{j\to\infty}\varepsilon_j=0$, the latter term on the right hand side also tends to 0. Consequently,

$$\lim_{j\to\infty}J_j\leq 0. \tag{4.26}$$

On the other hand (exactly as in $\mathbb{R}^n$), for every $x\in M$ and $X,Y\in T_xM$, we have the inequality

$$2^{2-n}|X-Y|^n\leq \langle |X|^{n-2}X-|Y|^{n-2}Y,X-Y\rangle.$$

Combining this inequality with (4.26) and using the properties of φ, it turns out that

$$\lim_{j\to\infty}\int_{A_{x_0}(\frac{r}{2})}\left(|\nabla_g u_j-\nabla_g u_G|^n+|u_j-u_G|^n\right){\rm d}v_g=0.$$

It remains to apply a covering argument as in Step 2 in order to prove (4.25).

Step 4: concluding the proof. By Step 3 (cf. (4.25)), we get in particular that the sequence $\{\nabla_g u_j\}$ converges (up to a subsequence) to $\nabla_g u_G$ almost everywhere on M. Since the sequence $\{|\nabla_g u_j|^{n-2}\nabla_g u_j\}$ is bounded in $L^{\frac{n}{n-1}}(M,TM)$, there exists $X_0 \in TM$ such that $|\nabla_g u_j|^{n-2}\nabla_g u_j \rightharpoonup X_0$ weakly in $L^{\frac{n}{n-1}}(M,TM)$. The a.e. convergence of the sequence $\{\nabla_g u_j\}$ to $\nabla_g u_G$ implies that X_0 should be precisely $|\nabla_g u_G|^{n-2}\nabla_g u_G$. Consequently,

$$|\nabla_g u_j|^{n-2}\nabla_g u_j \rightharpoonup |\nabla_g u_G|^{n-2}\nabla_g u_G \text{ weakly in } L^{\frac{n}{n-1}}(M,TM). \tag{4.27}$$

Let $w \in W_G^{1,n}(M)$ be arbitrarily fixed. By density, there exists a sequence $\{w_l\} \subset C_0^\infty(M)$ which converges to w in $\|\cdot\|_{0,1}$. By using w_l as a test-function in (4.9), due to relations (4.19), (4.27) and the fact that $\lim_{j\to\infty}\varepsilon_j = 0$, we have

$$\int_M (|\nabla_g u_G|^{n-2}\langle \nabla_g u_G, \nabla_g w_l\rangle + |u_G|^{n-2}u_G w_l)\mathrm{d}v_g - \int_M f(u_G)w_l \mathrm{d}v_g = 0 \text{ for all } l \in \mathbb{N}.$$

Letting now $l \to \infty$, it turns out that

$$\int_M (|\nabla_g u_G|^{n-2}\langle \nabla_g u_G, \nabla_g w\rangle + |u_G|^{n-2}u_G w)\mathrm{d}v_g - \int_M f(u_G)w \mathrm{d}v_g = 0,$$

which is nothing but $\langle \mathcal{E}_G'(u_G), w\rangle_* = 0$; thus, the arbitrariness of $w \in W_G^{1,n}(M)$ implies that $\mathcal{E}_G'(u_G) = 0$, concluding the proof. ☐

Since (M,g) is a Hadamard manifold, its injectivity radius is $+\infty$; thus, it costs no generality to consider in particular $\varepsilon_0 = 1$ and $\varepsilon := \frac{1}{j}$ $(j \in \mathbb{N}\setminus\{1\})$ in the function (3.1), introducing the rescaled Moser functions

$$m_j(x) := \frac{(\log j)^{\frac{n-1}{n}}}{\omega_{n-1}^{\frac{1}{n}}} u_{\frac{1}{j}}(x) = \frac{(\log j)^{\frac{n-1}{n}}}{\omega_{n-1}^{\frac{1}{n}}} \min\left\{\left(-\frac{\log d_g(x_0,x)}{\log j}\right)_+, 1\right\},\ x \in M. \tag{4.28}$$

The functions m_j are well-defined and $\mathrm{supp}(m_j) = \overline{B}_{x_0}(1)$ for every $j \in \mathbb{N}\setminus\{1\}$. Moreover, since $\mathrm{Fix}_M(G) = \{x_0\}$, it follows that the functions m_j are $G-$invariant for every $j \in \mathbb{N}\setminus\{1\}$; thus, $m_j \in W_G^{1,n}(M)$. Taking into account the computations from the proof of Proposition 1.1, it follows that

$$\|m_j\|_{0,1}^n = 1 + \mathcal{O}\left(\frac{1}{\log j}\right) \text{ as } j \to \infty. \tag{4.29}$$

Moreover, inspired by Adimurthi and Yang [3] and do Ó [20], we have

Lemma 4.4. *There exists $j_0 \in \mathbb{N}\setminus\{1\}$ such that*

$$\max_{s\geq 0} \mathcal{E}_G(sm_{j_0}) < \frac{1}{n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1},$$

where $\alpha_0 > 0$ is from hypothesis (f_1).

Proof. By contradiction, we assume that for every $j \in \mathbb{N}\setminus\{1\}$, we have

$$\max_{s\geq 0} \mathcal{E}_G(sm_j) \geq \frac{1}{n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}.$$

Since $\mathcal{E}_G(0)=0$ and $\mathcal{E}_G(sm_j)\to-\infty$ as $s\to\infty$ (cf. Lemma 4.2), there exists $s_j>0$ such that

$$\max_{s\geq 0}\mathcal{E}_G(sm_j)=\mathcal{E}_G(s_jm_j)=s_j^n\frac{\|m_j\|_{0,1}^n}{n}-\mathcal{F}(s_jm_j).$$

On one hand, since $\mathcal{F}\geq 0$, the above relations yield

$$s_j^n\|m_j\|_{0,1}^n\geq\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}. \tag{4.30}$$

Due to (4.29), the above inequality implies that

$$\liminf_{j\to\infty}s_j^n\geq\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}. \tag{4.31}$$

On the other hand, $s_j>0$ being an extremal point of $s\mapsto\mathcal{E}_G(sm_j)$, we also have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_G(sm_j)\Big|_{s=s_j}=0,$$

which is equivalent to

$$s_j^n\|m_j\|_{0,1}^n=\int_M f(s_jm_j)s_jm_j\mathrm{d}v_g \text{ for every } j\in\mathbb{N}\setminus\{1\}. \tag{4.32}$$

By (4.29), there exists $c_9>0$ such that for large $j\in\mathbb{N}$,

$$\|m_j\|_{0,1}^n\leq 1+\frac{c_9}{\log j}. \tag{4.33}$$

Fix

$$L_0>\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}\omega_n^{-1}e^{c_9\frac{n}{n-1}}.$$

By hypothesis (f_1), there exists $R_1>0$ such that

$$sf(s)e^{-\alpha_0 s^{\frac{n}{n-1}}}\geq L_0 \text{ for every } s\geq R_1. \tag{4.34}$$

Note that the sequence $\{s_j\}$ is bounded. Indeed, if we assume, up to a subsequence, that $\lim_{j\to\infty}s_j=\infty$, then for $j\in\mathbb{N}$ large enough, we have by (4.32) that

$$\begin{aligned}
\|m_j\|_{0,1}^n &\geq s_j^{-n}\int_{B_{x_0}(\frac{1}{j})}f(s_jm_j)s_jm_j\mathrm{d}v_g && [sf(s)\geq 0 \text{ for every } s\geq 0]\\
&\geq L_0s_j^{-n}\int_{B_{x_0}(\frac{1}{j})}e^{\alpha_0(s_jm_j)^{\frac{n}{n-1}}}\mathrm{d}v_g && [\text{see (4.34)}]\\
&= L_0s_j^{-n}e^{\alpha_0 s_j^{\frac{n}{n-1}}\omega_{n-1}^{-\frac{1}{n-1}}\log j}\mathrm{Vol}_g\left(B_{x_0}\left(\frac{1}{j}\right)\right) && [\text{see (4.28)}]\\
&\geq L_0\omega_ne^{n\left(\frac{\alpha_0}{\alpha_n}s_j^{\frac{n}{n-1}}-1\right)\log j-n\log s_j}. && [\text{see Proposition 2.1 (ii)}]
\end{aligned}$$

Letting $j\to\infty$, on account of (4.29) we arrive to a contradiction; thus, $\{s_j\}$ is bounded.

We claim that

$$\lim_{j\to\infty} s_j^n = \left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}. \tag{4.35}$$

By contradiction, due to (4.31), we assume that there exists $\varepsilon_0 > 0$ such that (up to a subsequence) for enough large $j \in \mathbb{N}$,

$$s_j^{\frac{n}{n-1}} > \frac{\alpha_n}{\alpha_0} + \varepsilon_0.$$

Note that for every $x \in B_{x_0}(\frac{1}{j})$, we have $s_j m_j(x) = s_j (\log j)^{\frac{n-1}{n}} \omega_{n-1}^{-1/n} \to \infty$ as $j \to \infty$. Therefore, for enough large $j \in \mathbb{N}$, relation (4.34) can be applied for $s = s_j m_j(x)$ with $x \in B_{x_0}(\frac{1}{j})$, obtaining in a similar manner as above that

$$s_j^n \|m_j\|_{0,1}^n \geq L_0 \omega_n e^{n\left(\frac{\alpha_0}{\alpha_n} s_j^{\frac{n}{n-1}} - 1\right)\log j}.$$

Consequently, the latter two inequalities, the boundedness of $\{s_j\}$ and (4.29) provide a contradiction once $j \to \infty$, which proves the validity of (4.35).

For every $j \in \mathbb{N} \setminus \{1\}$, let

$$A_j = \{x \in \overline{B}_{x_0}(1) : s_j m_j(x) \geq R_1\} \quad \text{and} \quad B_j = \overline{B}_{x_0}(1) \setminus A_j.$$

Due to (4.34), we have

$$\begin{aligned}
\int_M f(s_j m_j) s_j m_j \mathrm{d}v_g &= \int_{A_j} f(s_j m_j) s_j m_j \mathrm{d}v_g + \int_{B_j} f(s_j m_j) s_j m_j \mathrm{d}v_g \\
&\geq L_0 \int_{A_j} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g + \int_{B_j} f(s_j m_j) s_j m_j \mathrm{d}v_g \\
&= L_0 \int_{\overline{B}_{x_0}(1)} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g - L_0 \int_{B_j} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g \\
&\quad + \int_{B_j} f(s_j m_j) s_j m_j \mathrm{d}v_g.
\end{aligned} \tag{4.36}$$

Note that $s_j m_j \leq R_1$ in B_j, while $m_j \to 0$ and $\chi_{B_j} \to 1$ almost everywhere in $\overline{B}_{x_0}(1)$ as $j \to \infty$. Consequently, on one hand, by the Lebesgue dominated convergence theorem we have

$$\lim_{j\to\infty} \int_{B_j} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g = \int_{\overline{B}_{x_0}(1)} \mathrm{d}v_g = \mathrm{Vol}_g(\overline{B}_{x_0}(1)) \quad \text{and} \quad \lim_{j\to\infty} \int_{B_j} f(s_j m_j) s_j m_j \mathrm{d}v_g = 0.$$

On the other hand,

$$\int_{\overline{B}_{x_0}(1)} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g = \int_{\overline{B}_{x_0}(1) \setminus B_{x_0}(\frac{1}{j})} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g + \int_{B_{x_0}(\frac{1}{j})} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g =: I_j^1 + I_j^2.$$

Clearly, we have $I_j^1 \geq 0$, and for large $j \in \mathbb{N}$,

$$\begin{aligned} I_j^2 &= \int_{B_{x_0}(\frac{1}{j})} e^{\alpha_0 (s_j m_j)^{\frac{n}{n-1}}} \mathrm{d}v_g \\ &\geq \int_{B_{x_0}(\frac{1}{j})} e^{\alpha_n m_j^{\frac{n}{n-1}} \|m_j\|_{0,1}^{-\frac{n}{n-1}}} \mathrm{d}v_g = e^{n \log j \|m_j\|_{0,1}^{-\frac{n}{n-1}}} \mathrm{Vol}_g \left(B_{x_0}(\frac{1}{j}) \right) && \text{[see (4.30)]} \\ &\geq \omega_n j^{n\left(\|m_j\|_{0,1}^{-\frac{n}{n-1}} - 1 \right)} && \text{[see Proposition 2.1 (ii)]} \\ &\geq \omega_n j^{n\left(\left(1+\frac{c_9}{\log j}\right)^{-\frac{1}{n-1}} - 1 \right)}. && \text{[see (4.33)]} \end{aligned}$$

Therefore,

$$\liminf_{j\to\infty} I_j^2 \geq \omega_n \lim_{j\to\infty} j^{n\left(\left(1+\frac{c_9}{\log j}\right)^{-\frac{1}{n-1}} - 1 \right)} = \omega_n e^{-c_9 \frac{n}{n-1}}.$$

Putting in (4.32) the latter estimates together with relations (4.29), (4.35) and (4.36), it follows that

$$\left(\frac{\alpha_n}{\alpha_0} \right)^{n-1} \geq L_0 \omega_n e^{-c_9 \frac{n}{n-1}},$$

which contradicts the choice of L_0. The proof is complete. □

Proof of Theorem 1.4. Let $m_{j_0} \in W_G^{1,n}(M)$ be the Moser function which satisfies the conclusion of Lemma 4.4. By Lemma 4.2, the functional $\mathcal{E}_G : W_G^{1,n}(M) \to \mathbb{R}$ has the mountain pass geometry; in particular, if $e_0 = s_0 m_{j_0} \in W_G^{1,n}(M)$ with $s_0 > 0$ large enough, then $\mathcal{E}_G(e_0) < 0 = \mathcal{E}_G(0)$ and $\mathcal{E}_G(u) \geq \tilde{\delta} > 0$ for every $u \in W_G^{1,n}(M)$ with $\|u\|_{0,1} = \tilde{r}$, where $\tilde{r} < \|e_0\|$. By using the mountain pass lemma for $\mathcal{E}_G$ without the Palais-Smale compactness condition, see e.g. Brezis and Nirenberg [8, p. 943], there exists a sequence $\{u_j\} \subset W_G^{1,n}(M)$ such that

$$\mathcal{E}_G(u_j) \to c \ \text{ and } \ \mathcal{E}_G'(u_j) \to 0 \text{ in } W_G^{1,n}(M)^*, \tag{4.37}$$

where

$$c = \inf_{\lambda\in\Lambda} \max_{s\in[0,1]} \mathcal{E}_G(\lambda(s)) \geq \tilde{\delta},$$

and $\Lambda = \{\lambda \in C([0,1], W_G^{1,n}(M)) : \lambda(0) = 0,\ \lambda(1) = e_0\}$. According to Lemma 4.3, there exists $u_G \in W_G^{1,n}(M)$ such that, up to a subsequence,

$$\lim_{j\to\infty} \mathcal{F}(u_j) = \mathcal{F}(u_G), \tag{4.38}$$

$u_j \to u_G$ strongly in $L^p(M)$ for every $p \in (n, \infty)$, and u_G is a critical point of $\mathcal{E}_G$. The latter fact with Lemma 4.1 shows that u_G is a non-negative $G-$invariant weak solution of $(\mathcal{P})$.

It remains to prove that $u_G \neq 0$. By contradiction, if $u_G = 0$, relations (4.37) and (4.38) imply on one hand that

$$\lim_{j\to\infty} \|u_j\|_{0,1}^n = nc \geq n\tilde{\delta} > 0. \tag{4.39}$$

On the other hand, if we apply u_j as a test-function in $\mathcal{E}_G'(u_j) \to 0$, one has $\lim_{j\to\infty}\langle \mathcal{E}_G'(u_j), u_j\rangle_* = 0$, i.e.,

$$\lim_{j\to\infty}\left(\|u_j\|_{0,1}^n - \int_M f(u_j)u_j \mathrm{d}v_g\right) = 0. \tag{4.40}$$

By the definition of the minimax value c, it follows by Lemma 4.4 that

$$c \leq \max_{s\in[0,1]} \mathcal{E}_G(se_0) \leq \max_{s\geq 0} \mathcal{E}_G(sm_{j_0}) < \frac{1}{n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}.$$

This estimate and (4.39) guarantee the existence of $q > n$ such that for every large $j \in \mathbb{N}$,

$$\frac{q}{q-1}\|u_j\|_{0,1}^{\frac{n}{n-1}} < \frac{\alpha_n}{\alpha_0}.$$

Relations (4.2), (4.3), the Hölder's inequality and the latter relation imply that for large $j \in \mathbb{N}$,

$$\begin{aligned} 0 \leq \int_M f(u_j)u_j \mathrm{d}v_g &\leq c_0 \int_M |u_j|^\gamma \mathrm{d}v_g + c_0 \int_M |u_j|\Phi_n(\alpha_0|u_j|^{\frac{n}{n-1}})\mathrm{d}v_g \\ &\leq c_0\|u_j\|_{L^\gamma(M)}^\gamma + c_0\|u_j\|_{L^q(M)}\left(\int_M \Phi_n\left(\alpha_0\frac{q}{q-1}|u_j|^{\frac{n}{n-1}}\right)\mathrm{d}v_g\right)^{1-\frac{1}{q}} \\ &\leq c_0\|u_j\|_{L^\gamma(M)}^\gamma + c_0\|u_j\|_{L^q(M)}\left(S^0_{\alpha_n,1}(M,g)\right)^{1-\frac{1}{q}}. \end{aligned}$$

Note that $S^0_{\alpha_n,1}(M,g) < \infty$, cf. Yang, Su and Kong [46, Theorem 1.2]. Moreover, since $\gamma, q > n$ and $\lim_{j\to\infty}\|u_j\|_{L^p(M)} = 0$ for every $p \in (n,\infty)$, it follows from the last estimate that

$$\lim_{j\to\infty}\int_M f(u_j)u_j \mathrm{d}v_g = 0.$$

This limit and relations (4.39) and (4.40) provide a contradiction. Therefore, $u_G \neq 0$, which completes the proof. □

We conclude the paper by presenting some possible scenarios where Theorem 1.4 can be applied.

Example 4.1. [*Euclidean case*] If $(M,g) = (\mathbb{R}^n, g_{\text{euc}})$ is the usual Euclidean space, Theorem 1.4 can be applied for $x_0 = 0$ and $G = \mathsf{SO}(n_1,\mathbb{R}) \times ... \times \mathsf{SO}(n_l,\mathbb{R})$ with $n_j \geq 2$, $j = 1,...,l$ and $n_1 + ... + n_l = n$, where $\mathsf{SO}(m,\mathbb{R})$ is the special orthogonal group in $\mathbb{R}^m$. Indeed, we have $\text{Fix}_{\mathbb{R}^n}(G) = \{0\}$ and $O_G^x = |x_{n_1}|\mathbb{S}^{n_1-1} \times ... \times |x_{n_l}|\mathbb{S}^{n_l-1}$ for each $x = (x_{n_1},...,x_{n_l}) \in \mathbb{R}^{n_1}\times ... \times \mathbb{R}^{n_l} \setminus \{0\}$. □

Example 4.2. [*Hyperbolic case*] For the hyperbolic space we use the Poincaré ball model $\mathbb{H}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the Riemannian metric $g_{\text{hyp}}(x) = (g_{ij}(x))_{i,j=1,...,n} = \frac{4}{(1-|x|^2)^2}\delta_{ij}$. It is well known that $(\mathbb{H}^n, g_{\text{hyp}})$ is a homogeneous Hadamard manifold with constant sectional curvature -1. Theorem 1.4 can be applied with the same choice for x_0 and G as in Example 4.1. □

Example 4.3. [*Symmetric positive definite matrices*] Let $\mathrm{Sym}(n,\mathbb{R})$ be the set of symmetric $n\times n$ matrices with real values, $\mathrm{P}(n,\mathbb{R})\subset \mathrm{Sym}(n,\mathbb{R})$ be the $\frac{n(n+1)}{2}-$dimensional cone of symmetric positive definite matrices, and $\mathrm{P}(n,\mathbb{R})_1$ be the subspace of matrices in $\mathrm{P}(n,\mathbb{R})$ with determinant one. The set $\mathrm{P}(n,\mathbb{R})$ is endowed with the scalar product

$$\langle\langle U,V\rangle\rangle_X=\mathrm{Tr}(X^{-1}VX^{-1}U)\ \text{ for all }\ X\in \mathrm{P}(n,\mathbb{R}),\ U,V\in T_X(\mathrm{P}(n,\mathbb{R}))\simeq \mathrm{Sym}(n,\mathbb{R}),$$

where $\mathrm{Tr}(Y)$ denotes the trace of $Y\in \mathrm{Sym}(n,\mathbb{R})$, and let us denote by $d_H:\mathrm{P}(n,\mathbb{R})\times \mathrm{P}(n,\mathbb{R})\to\mathbb{R}$ the induced metric function. The pair $(\mathrm{P}(n,\mathbb{R}),\langle\langle\cdot,\cdot\rangle\rangle)$ is a Hadamard manifold, see Lang [33, Chapter XII]. Note that $\mathrm{P}(n,\mathbb{R})_1$ is a convex totally geodesic submanifold of $\mathrm{P}(n,\mathbb{R})$ and the special linear group $\mathsf{SL}(n,\mathbb{R})$ leaves $\mathrm{P}(n,\mathbb{R})_1$ invariant and acts transitively on it; thus $(\mathrm{P}(n,\mathbb{R})_1,\langle\langle\cdot,\cdot\rangle\rangle)$ is itself a homogeneous Hadamard manifold, see Bridson and Haefliger [9, Chapter II.10]. Moreover, for every $\sigma\in\mathsf{SL}(n,\mathbb{R})$, the map $[\sigma]:\mathrm{P}(n,\mathbb{R})_1\to \mathrm{P}(n,\mathbb{R})_1$ defined by $[\sigma](X)=\sigma X\sigma^t$, is an isometry; here, σ^t denotes the transpose of σ.

Let $G=\mathsf{SO}(n,\mathbb{R})$. One can prove that $\mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G)=\{I_n\}$, where I_n is the identity matrix. First, it is clear that $I_n\in \mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G)$; indeed, for every $\sigma\in G$ we have $[\sigma](I_n)=\sigma I_n\sigma^t=\sigma\sigma^t=I_n$. Second, if $X_0\in \mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G)$, then it turns out that $\sigma X_0=X_0\sigma$ for every $\sigma\in G$. By using elementary matrices from G, the latter relation implies that $X_0=cI_n$ for some $c\in\mathbb{R}$. Since $X_0\in \mathrm{P}(n,\mathbb{R})_1$, we necessarily have $c=1$. Moreover, the orbit O_G^X of the matrix $X\in \mathrm{P}(n,\mathbb{R})_1\setminus\{I_n\}$ under the action of G is the geodesic sphere in $\mathrm{P}(n,\mathbb{R})_1$ with center I_n and radius $d_H(I_n,X)$; in particular, $\mathrm{Card}(O_G^X)=\infty$. Indeed, for every $\sigma\in G$, since $[\sigma]$ is an isometry on $\mathrm{P}(n,\mathbb{R})_1$, it follows that

$$d_H^2(I_n,[\sigma](X)) \;=\; d_H^2([\sigma](I_n),[\sigma](X))=d_H^2(I_n,X).$$

Consequently, Theorem 1.4 is applicable on $\mathrm{P}(n,\mathbb{R})_1$ with the choices $x_0=I_n$ and $G=\mathsf{SO}(n,\mathbb{R})$, respectively.

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